

On the arboreal structure of right-angled Artin groups

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Abstract

The present article continues the study of median groups initiated in [6, 9, 10]. Some classes of median groups are introduced and investigated, with a stress upon the class of the so called *A-groups* which contains as remarkable subclasses the *lattice ordered groups* and the *right-angled Artin groups*. Some general results concerning *A-groups* are applied to a systematic study of the arboreal structure of right-angled Artin groups. Structure theorems for foldings, directions, quasidirections and centralizers are proved.

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1 Introduction

In his paper [9], an improved version of the preprints [3, 4], the author applied the theory of generalized trees (median sets) developed in [2, 7] to elaborate a general theory of median (or arboreal) groups and use it to the investigation of a remarkable class of groups called *partially commutative Artin-Coxeter groups*.

Recall that by a *median* or *arboreal group* we mean a group G endowed with a ternary operation $Y : G^3 \longrightarrow G$ making it a *median set* or *generalized tree* such that $uY(x, y, z) = Y(ux, uy, uz)$ for all $u, x, y, z \in G$.

Equivalently, according to [9, Proposition 2.2.1.], by a median group we can also understand a group G endowed with a meet-semilattice operation \cap , with associated order \subset , satisfying the following three axioms :

$$(1) \forall x \in X, 1 \subset x$$

- (2) $\forall x, y, z \in X, x \subset y \text{ and } y \subset z \implies z^{-1}y \subset z^{-1}x$
 (3) $\forall x, y \in X, x^{-1}(x \cap y) \subset x^{-1}y$

In a median group G , an ordered pair $(x, y) \in G^2$ is said to be *reduced* (write $xy = x \bullet y$) if $x \subset xy$, i.e. $x^{-1} \cap y = 1$. For all $x, y \in G$, $x \cap y$ is the unique element $z \in G$ satisfying $x = z \bullet (z^{-1}x)$, $y = z \bullet (z^{-1}y)$, and $x^{-1}y = (x^{-1}z) \bullet (z^{-1}y)$. Notice also that $xy \cap xz = x(y \cap z)$ (in particular, $xy \subset xz \iff y \subset z$) provided the pairs (x, y) and (x, z) are reduced. In other words, $x^{-1}y \cap x^{-1}z = x^{-1}(y \cap z)$ whenever $x \subset y$ and $x \subset z$.

The elements x and y of a median group G are said to be *orthogonal* (write $x \perp y$) if $x \cap y = 1$ and there exists the join $x \cup y$ (write $x \cup y \neq \infty$). According to [9, 2.2.], a median group G is said to be a \perp -group if $\forall x, y \in G, x \perp y \implies x \cup y = xy$, in particular, $xy = yx$.

Notice that in a \perp -group G , for all $x, y \in G$, $x \cup y \neq \infty \implies x \cup y = x(x \cap y)^{-1}y = y(x \cap y)^{-1}x$. In particular, a subgroup H of G is median, i.e. $x, y, z \in H \implies Y(x, y, z) \in H$, iff $x, y \in H \implies x \cap y \in H$, while H is convex, i.e. $x, y \in H \implies \forall z \in G, Y(x, y, z) \in H$, iff $\forall x \in H, y \in G, y \subset x \implies y \in H$. By [10, Corollary 2.8.], for any \perp -group G there exists a canonical simple transitive action of G on a subdirect product of locally linear median sets.

Given a group G and a set $S \subseteq G$ of generators such that $1 \notin S$ and $S_1 := S \cap S^{-1} = \{s \in S \mid s^2 = 1 \text{ holds in } G\}$, it turns out by [9, Theorem 2.4.1.] that the partial order on G defined by $x \subset y \iff l(x) + l(x^{-1}y) = l(y)$, where $l : G \longrightarrow \mathbb{N}$ denotes the canonical length function on (G, S) , makes G a \perp -group if and only if (G, S) is a *partially commutative Artin-Coxeter system*, i.e. the group G admits the presentation

$$G = \langle S; s^2 = 1 \text{ for } s \in S_1, [s, t] = 1 \text{ for } s, t \in S, s \neq t, st = ts \text{ holds in } G \rangle$$

Thus the partially commutative Artin-Coxeter groups are identified with the simplicial \perp -groups.

The present paper, an improved version of the preprint [5], is devoted to a systematic study of the arboreal structure of the systems (G, S) above which satisfy the additional restrictive assumption that $S \cap S^{-1} = \emptyset$. Such systems were introduced by Baudisch in [13, 14] under the name of *semi-free groups*, and extensively studied in the last years under various names (*right-angled Artin groups*, *free partially commutative groups*, *graph groups*) by people working in combinatorial and geometric group theory, associative algebras, computer science (see for instance the long bibliography to the survey article [17]). Their nice properties were exploited by Bestvina and Brady [15] in their construction of examples of groups which are of type (FP) but are not finitely presented, as well as by Crisp and Paris [18] in their proof of a conjecture of Tits on the subgroup generated by the squares of the generators of an Artin group. According to [19], the finitely generated right-angled Artin groups (moreover, the weakly partially commutative Artin-Coxeter groups as defined in [9, 1.1.]) are linear and hence equationally noetherian.

The outline of the paper is as follows. Some notions and basic facts from [2] on *congruences* and *quasidirections* on median sets are recalled in Section 2. Some classes of median groups are introduced and investigated in Section 3. Amongst them, the class of the so called *A-groups* contains as remarkable subclasses the *l-groups*, not

necessarily commutative, and the right-angled Artin groups. The *cyclically reduced elements* of A -groups are studied in Section 4, while Sections 5 and 6 are devoted to the main properties of the preorders \preceq_w and the foldings φ_w naturally associated to any element w of an A -group.

The general theory of A -groups is further applied in the last two sections of the paper to the particular case of right-angled Artin groups. One shows in Section 7 that in a right-angled Artin group G the preorders \preceq determine quasidirections \bullet_w which are described as limits of sequences of operators $w^n \varphi_w$ for $n \rightarrow \infty$.

The main results of the paper contained in Section 8 provide descriptions of the quasidirections \bullet_w , the foldings φ_w and the centralizers $Z_G(w)$ in terms of the corresponding invariants \bullet_p , φ_p and $Z_G(p)$, where p ranges over a finite set $\text{Prim}(w)$ of *primitive elements* canonically associated to any element w of a right-angled Artin group G .

2 Congruences and quasidirections on median sets

In this section having a preliminary character we recall some notions and basic facts from [2] which will be used later.

By a *median set* or *generalized tree* we understand a set X endowed with a ternary operation $Y : X^3 \rightarrow X$, called *median*, satisfying the following equational axioms:

- (i) *Symmetry* : $Y(x, y, z) = Y(y, x, z) = Y(x, z, y)$,
- (ii) *Absorptive law* : $Y(x, y, x) = x$, and
- (iii) *Selfdistributive law* : $Y(Y(x, y, z), u, v) = Y(Y(x, u, v), y, Y(z, u, v))$.

In a median set X , for any $a \in X$, the binary operation $(x, y) \mapsto x \vee_a y := Y(x, a, y)$ makes X a join-semilattice with the last element a ; let \leq_a denote the corresponding partial order.

A subset I of a median set X is said to be *convex* if $Y(x, y, z) \in I$ whenever $x, y \in I, z \in X$. As the intersection of an arbitrary family of convex subsets is also convex, we may speak on the *convex closure* of any subset A of X and denote it by $[A]$. In particular, set $[a, b] =: [\{a, b\}]$ for $a, b \in X$.

By a *cell* of a median set X we mean a convex subset C of X for which there are $a, b \in X$ such that $C = [a, b]$. Given a cell C , every element $a \in X$ for which there exists $b \in X$ such that $C = [a, b]$ is called an *end* of C . The (non-empty) subset of all ends of a cell C , denoted by ∂C and called the *boundary* of C , is a median subset of C , and the mapping \neg assigning to each $a \in \partial C$ the unique end $\neg a$ of C for which $C = [a, \neg a]$ is an involutory automorphism of the median set ∂C . Note also that for a given $a \in \partial C$, the cell C becomes a bounded distributive lattice with respect to the order \leq_a , with the join \vee_a , the meet \neg_a , the last element a , and the least element $\neg a$, while its boundary ∂C is identified with the boolean subalgebra consisting of those elements which have (unique) complements.

The median set X is called *locally boolean*, resp. *locally linear*, if $C = \partial C$ for every cell C of X , resp. $\partial[x, y] = \{x, y\}$ for all $x, y \in X$. X is called *simplicial* (or

discrete or *locally finite*) if every cell of X has finitely many elements. A graph-theoretic description for simplicial median sets is given in [11, Lemma 7.1., Proposition 7.3.] In particular, the *trees*, i.e. the acyclic connected graphs, are naturally identified with the simplicial locally linear median sets.

Note that the convex closure of a finite subset of a simplicial median set is finite too, and hence every non-empty convex subset is retractible. To any simplicial median set X one assigns an integer-valued "distance" function $d : X \times X \rightarrow \mathbb{N}$, where for $x, y \in X$, $d(x, y)$ is the length of some (of any) maximal chain in the finite distributive lattice $([x, y], \leq)$. With respect to d , X becomes a \mathbb{Z} -metric space such that for all $x, y \in X$, $[x, y] = \{z \in X \mid d(x, z) + d(z, y) = d(x, y)\}$, and the mapping $[x, y] \rightarrow [0, d(x, y)], z \mapsto d(x, z)$, induced by d , is onto. In particular, X is a tree iff for all $x, y \in X$, the mapping above is bijective.

2.1 Congruences on median sets

Given a median set X , a *congruence* on X is an equivalence relation ρ on X which is compatible with the median Y , i.e. for all $a, b, x, y \in X$,

$$(x, y) \in \rho \implies (Y(a, b, x), Y(a, b, y)) \in \rho.$$

The congruences on X form a lattice $\text{Cong}(X)$ with a least and a last element under the inclusion of relations. Moreover, according to [2, Proposition 1.6.1.], the lattice $\text{Cong}(X)$ is a Heyting algebra (in particular, a bounded distributive lattice), i.e. for every pair (ρ, γ) of congruences on X there exists a unique congruence $\mu := \rho \rightarrow \gamma$ subject to $\theta \subseteq \mu \iff \theta \cap \rho \subseteq \gamma$ for all $\theta \in \text{Cong}(X)$, namely the congruence

$$\mu = \{(a, b) \in X \times X \mid \forall x, y \in [a, b], (x, y) \in \rho \implies (x, y) \in \gamma\}.$$

In particular, for $\gamma = \Delta$, the equality on X , we obtain the *negation* of ρ

$$\neg \rho := \rho \rightarrow \Delta = \{(a, b) \in X \times X \mid \rho|_{[a, b]} = \Delta|_{[a, b]}\}.$$

By [2, Corollary 1.6.2.], $\text{Cong}(X)$ is a boolean algebra provided the median set X is simplicial.

Given a simplicial median set X and a congruence \sim on X , let \equiv denote the negation (the complement) $\neg \sim$ of \sim in the boolean algebra $\text{Cong}(X)$. For every $a \in X$, set $\tilde{a} = \{x \in X \mid x \sim a\}$, $\bar{\bar{a}} = \{x \in X \mid x \equiv a\}$, and let φ_a , resp. ψ_a , denote the folding induced by the (retractible) convex subset \tilde{a} , resp. $\bar{\bar{a}}$. Thus for all $x \in X$, $[a, x] \cap \tilde{a} = [a, \varphi_a(x)]$ and $[a, x] \cap \bar{\bar{a}} = [a, \psi_a(x)]$.

Lemma 2.1. *Let \sim be a congruence on a simplicial median set X , with its negation \equiv . Then, the following assertions are equivalent.*

- (1) *For all $a, b \in X$, the intersection $\tilde{a} \cap \bar{\bar{b}}$ is nonempty.*
- (2) *For all $a, b \in X$, $\varphi_a(b) \equiv b$, i.e. for all $a \in X$, the embedding $\tilde{a} \longrightarrow X$ induces a median set isomorphism $\tilde{a} \longrightarrow X/\equiv$.*

- (3) For all $a, b \in X$, $\psi_a(b) \sim b$, i.e. for all $a \in X$, the embedding $\bar{\bar{a}} \longrightarrow X$ induces a median set isomorphism $\bar{\bar{a}} \longrightarrow X / \sim$.
- (4) For all $a, b \in X$, $\varphi_a(b) = \psi_b(a)$.
- (5) For all $a, b \in X$, $[a, b] = [\varphi_a(b), \psi_a(b)]$.
- (6) For every quasi-linear cell $[a, b]$ (i.e. $\partial[a, b] = \{a, b\}$), either $a \sim b$ or $a \equiv b$.
- (7) For every cell $[a, b]$ with three elements, either $a \sim b$ or $a \equiv b$.

Proof. The implications (4) \implies (2), (4) \implies (3), (2) \implies (1), (3) \implies (1) and (6) \implies (7) are trivial.

(1) \implies (4). If $\tilde{a} \cap \bar{\bar{b}}$ is non-empty, then obviously $\tilde{a} \cap \bar{\bar{b}} = \{c\}$ is a singleton. Note also that $c = Y(c, c, b) \sim Y(a, c, b)$ and $c = Y(a, c, c) \equiv Y(a, c, b)$, therefore $c = Y(a, c, b)$, i.e. $c \in [a, b]$. Consequently, $\{c\} = [a, b] \cap \tilde{a} \cap \bar{\bar{b}} = [a, \varphi_a(b)] \cap [b, \psi_b(a)] = [Y(a, \varphi_a(b), b) = \varphi_a(b), Y(a, \varphi_a(b), \psi_b(a)) = Y(a, b, \psi_b(a)) = \psi_b(a), Y(\varphi_a(b), b, \psi_b(a))]$, and hence $c = \varphi_a(b) = \psi_b(a)$ as required.

(4) \implies (5). The inclusion $[\varphi_a(b), \psi_a(b)] \subseteq [a, b]$ is obvious. On the other hand, $Y(a, \varphi_a(b), \psi_a(b)) \sim Y(a, a, \psi_a(b)) = a$, and $Y(a, \varphi_a(b), \psi_a(b)) \equiv Y(a, \varphi_a(b), a) = a$, therefore $Y(a, \varphi_a(b), \psi_a(b)) = a$, i.e. $a \in [\varphi_a(b), \psi_a(b)]$. By symmetry, we get $b \in [\varphi_b(a), \psi_b(a)] = [\psi_a(b), \varphi_a(b)]$ (by assumption). Thus $[a, b] \subseteq [\varphi_a(b), \psi_a(b)]$ as desired.

(5) \implies (6). Since the cell $[a, b] = [\varphi_a(b), \psi_a(b)]$ is assumed to be quasilinear, it follows that either $\varphi_a(b) = b$, i.e. $a \sim b$, or $\psi_a(b) = b$, i.e. $a \equiv b$.

(7) \implies (2). As $\varphi_a = \varphi_{\varphi_a(b)}$, we may assume without loss that $\varphi_a(b) = a$, so we have to show that $a \equiv b$. We argue by induction on the "distance" $d := d(a, b)$. Since the cases $d = 0$ and $d = 1$ are trivial, we may assume $d \geq 2$. Let $c \in [a, b]$ be such that $d(c, b) = 2$, and let $e \in [c, b] \setminus \{c, b\}$. Since $\varphi_a(e) = \varphi_a(Y(a, e, b)) = Y(\varphi_a(a), e, \varphi_a(b)) = Y(a, e, a) = a$ and $d(a, e) = d - 1 < d$, it follows by the induction hypothesis that $a \equiv e$, therefore $c \equiv a \equiv e$, as $c \in [a, e]$. We distinguish the following two cases:

Case (i) : The cell $[c, b]$ has three elements, i.e. $[c, b] = \{c, e, b\}$. By assumption either $c \equiv b$ or $c \sim b$. In the former case we get $a \equiv b$, as required, while in the latter case it follows that $c \sim e$ as $e \in [c, b]$. Since, on the other hand, $c \equiv e$, we get $c = e$, i.e. a contradiction.

Case (ii) : The cell $[c, b]$ has four elements, say $[c, b] = [e, f] = \{c, b, e, f\}$. As we already know that $e \equiv a \equiv f$, we get $a \equiv b$ since $b \in [e, f]$. \square

Corollary 2.2. *Given two complementary congruences \sim and \equiv on a simplicial median set X , assume that for each cell $[a, b]$ with three elements either $a \sim b$ or $a \equiv b$. For all $a \in X$, let φ_a , resp. ψ_a , denote the folding of X induced by the convex subset \tilde{a} , resp. $\bar{\bar{a}}$. Then, for all $a \in X$, the median set morphism $X \longrightarrow \tilde{a} \times \bar{\bar{a}}, x \mapsto (\varphi_a(x), \psi_a(x))$ is an isomorphism, whose inverse sends a pair $(x, y) \in \tilde{a} \times \bar{\bar{a}}$ to $\psi_x(y) = \varphi_y(x)$.*

2.2 Directions and quasidirections on median sets

By a *quasidirection* on a median set X we understand a binary operation \bullet on X satisfying the following four conditions :

- i) (X, \bullet) is a *band*, i.e. a semigroup in which all elements are idempotent,

- ii) $a \bullet b \bullet c = a \bullet c \bullet b$ for all $a, b, c \in X$,
- iii) for all $a \in X$, the left translation $X \rightarrow X, x \mapsto a \bullet x$ is a folding, i.e. $a \bullet Y(x, y, z) = Y(a \bullet x, y, a \bullet z)$ for all $x, y, z \in X$, and
- iv) for all $x, y, z \in X, Y(x, y, z) \bullet x = Y(x, y, z \bullet x)$.

Moreover, by [2, Lemma 3.3.], a stronger form of iv), the symmetrical version of iii), is also satisfied :

- iii)' for all $a \in X$, the right translation $X \rightarrow X, x \mapsto x \bullet a$ is a folding of X .

A quasidirection \bullet on X is said to be a *direction* if the band (X, \bullet) is a semilattice, i.e. $x \bullet y = y \bullet x$ for all $x, y \in X$. In this case, iv) becomes superfluous. Any element a of X determines a direction \bigvee_a on X given by $x \bigvee_a y := Y(x, a, y)$. Such directions are called *internal* or *closed*, while the other ones, if exist, are called *external* or *open*.

Call *trivial* the quasidirection defined by the rule $x \bullet y = x$.

A median set X endowed with a quasidirection, resp. a direction, is said to be *quasidirected*, resp. *directed*.

According to [2, Proposition 3.7.], the mapping assigning to a binary operation \bullet on the median set X the binary relation $a \leq_b b \iff b \bullet a = b$ maps bijectively the set of quasidirections on X onto the set of the preorders \preceq on X satisfying

- i) \preceq is compatible with the median of X , i.e. $\forall a, b, x, y \in X, x \preceq y \implies Y(a, b, x) \preceq Y(a, b, y)$; let \sim denote the congruence induced by the preorder \preceq , and let \equiv be its negation in the Heyting algebra $\text{Cong}(X)$;
- ii) for all $a, b \in X$ there exists $c \in X$ such that $a \preceq c, b \preceq c$, and $a \equiv c$.

The inverse of the bijection above sends a preorder \preceq as above to the quasidirection \bullet given by $a \bullet b = Y(a, b, c)$ for some (for all) $c \in X$ subject to ii). Note also that $a \equiv b \iff a \bullet b = b \bullet a$.

The bijection above induces by restriction a bijection of the set of directions on X onto the set of the orders of X which are compatible with the median of X such that any pair (a, b) of elements in X is bounded above.

According to [2, Corollary 3.5.], the canonical embedding $X \rightarrow X/\sim \times X/\equiv$ yields a representation of the quasidirected median set (X, \bullet) as a subdirect product of a pair consisting of a directed median set X/\sim and a trivially quasidirected median set X/\equiv , in such a way that the product $X/\sim \times X/\equiv$ is the convex closure of its median subset X .

Given a median set X , let $\text{Dir}(X)$, $\text{Fold}(X)$ and $\text{Qdir}(X)$ respectively denote the set of directions, of foldings and of quasidirections on X . By [2, Sections 8, 10], $\text{Qdir}(X)$ becomes a directed median set with the median $(q_1, q_2, q_3) \mapsto Y(q_1, q_2, q_3)$ given by $a \bullet_{Y(q_1, q_2, q_3)} b = Y(a \bullet_{q_1} b, a \bullet_{q_1} b, a \bullet_{q_3} b)$, and the direction induced by the order $q_1 \leq q_2$ iff the preorder \preceq_{q_2} associated to q_2 is finer than \preceq_{q_1} . The subset $\text{Dir}(X)$ is a median subset of $\text{Qdir}(X)$ consisting of the minimal elements under the order \leq on $\text{Qdir}(X)$, while the injective mapping $X \rightarrow \text{Dir}(X), a \mapsto \bigvee_a$, identifies X with a convex subset of $\text{Dir}(X)$. On the other hand, by [2, Proposition 9.1.], $\text{Fold}(X)$ has a canonical structure of directed median set with the median defined by $Y(\varphi_1, \varphi_2, \varphi_3)(x) = Y(\varphi_1(x), \varphi_2(x), \varphi_3(x))$, and the direction induced by the order $\varphi \leq \Psi$ iff $\varphi(X) \subseteq \Psi(X)$. Note that the injective mapping $X \rightarrow \text{Fold}(X), a \mapsto (x \mapsto a)$

identifies X with a median subset of $Fold(X)$.

According to [2, Theorem 9.3.], the mapping $\alpha : Dir(Fold(X)) \longrightarrow Fold(Dir(X))$, given by $a \underset{\alpha(d)(D)}{\vee} b = (a \underset{d}{\vee} b)(a \underset{D}{\vee} b) =$ the value in $a \underset{D}{\vee} b$ of the folding $a \underset{d}{\vee} b$, for $d \in Dir(Fold(X)), D \in Dir(X), a, b \in X$, is an isomorphism of median sets, while by [2, Theorem 10.1.], the map $\gamma : Fold(Dir(X)) \longrightarrow Qdir(X)$, given by $a \underset{\gamma(\varphi)}{\bullet} b = a \underset{\varphi(a)}{\vee} b$ for $\varphi \in Fold(Dir(X)), a, b \in X$, is an isomorphism of directed median sets.

Given a simplicial median set X and a preorder \preceq on X which is compatible with the median Y , let \sim denote the congruence induced by \preceq , with its complement \equiv in the boolean algebra $Cong(X)$. Recall that $x \equiv y \iff \forall u, v \in [x, y], u \sim v \implies u = v$. Assume that any pair (a, b) of elements in X is bounded above with respect to the preorder \preceq , i.e. there exists $c \in X$ such that $a \preceq c$ and $b \preceq c$. For $a, b \in X$, set $U_{a,b} = \{x \in [a, b] \mid a \preceq x \text{ and } b \preceq x\}$. By assumption, the finite set $U_{a,b}$ is nonempty. Indeed, if c is a common upper bound of the elements a and b , then $a = Y(a, b, a) \preceq Y(a, b, c)$, and $b = Y(a, b, b) \preceq Y(a, b, c)$, therefore $Y(a, b, c) \in U_{a,b}$. Define the binary operation \bullet on X by $a \bullet b = \underset{a}{\vee} U_{a,b}$.

With the notation and the data above we have

Lemma 2.3. (1) $U_{a,b} = [a \bullet b, b \bullet a]$.

(2) $a \bullet b \sim b \bullet a$,

(3) $a \underset{b}{\leq} a \bullet b \underset{b}{\leq} b \bullet a \underset{b}{\leq} b$.

(4) $a \preceq b \iff b \bullet a = b$.

Proof. (1) Since by assumption the preorder \preceq is compatible with the median Y , it follows that the nonempty set $\{x \in X \mid a \preceq x \text{ and } b \preceq x\}$ is a convex subset of X . Consequently, its image $U_{a,b}$ through the folding $X \longrightarrow X, x \mapsto Y(a, b, x)$, is a convex subset of the cell $[a, b]$. In particular, the cell $[a \bullet b, b \bullet a]$ is contained in $U_{a,b}$. On the other hand, for any $c \in U_{a,b}$ it follows by definition that $a \bullet b \in [a, c]$ and $b \bullet a \in [b, c]$. Consequently, $c \in [a \bullet b, b \bullet a]$ since otherwise, by [7, Corollary 5.2.2.], there exists a prime convex subset P of X such that $a \bullet b \in P, b \bullet a \in P$, and $c \notin P$, therefore $c \in [a, b] \subseteq P$, a contradiction.

The statements (2), (3) and (4) are obvious. \square

The next lemma provides a characterization of those preorders on a simplicial median set which induce quasidirections.

Lemma 2.4. *Let \preceq be a preorder on a simplicial median set X which is compatible with the median Y , such that any pair (a, b) of elements of X is bounded above with respect to \preceq . With the notation above, the following assertions are equivalent.*

(1) *The binary operation \bullet induced by the preorder \preceq is a quasidirection on X .*

(2) $\forall a, b \in X, a \bullet b = b \implies a \equiv b$.

(3) *For all $a, b, c \in X$ such that $[a, b] = \{a, c, b\}$ and $c \notin \{a, b\}, c \preceq b \implies a \preceq b$.*

Proof. (1) \implies (3). Let $a, b, c \in X$ be such that $[a, b] = \{a, c, b\}, c \notin \{a, b\}$ and $c \preceq b$. Assuming that $a \not\preceq b$ it follows that $U_{a,b} = \{a\}$, i.e. $a \bullet b = b \bullet a = a$. In particular, $b \sim c$ since $c \preceq b, c \in [b, a]$ and $b \preceq a$. Since by assumption the binary operation \bullet is a quasidirection on X , we get $a \equiv b$, contrary to $b \sim c, b \neq c$.

(3) \implies (2). Let $a, b \in X$ be such that $a \bullet b = b$, i.e. $U_{a,b} = \{b\}$. To show that $a \equiv b$ we argue by induction on the distance $d := d(a, b)$. Since the cases $d = 0$ and $d = 1$ are trivial, we may assume $d \geq 2$. Let $c \in [a, b]$ be such that $d(a, c) = 2$, and let $x \in [a, c] \setminus \{a, c\}$. It follows that $U_{x,b} = \{b\}$, i.e. $x \bullet b = b$, since $y \in U_{x,b} \implies a \preceq b \preceq y$, and hence $y \in U_{a,b} = \{b\}$. As $d(x, b) = d - 1 < d$, it follows by the induction hypothesis that $x \equiv b$. We distinguish the following two cases :

Case (i). The cell $[a, c]$ has four elements, say $[a, c] = [x, y] = \{a, c, x, y\}$. As we already know that $x \equiv b$ and $y \equiv b$, it follows that $a \equiv b$ since $a \in [x, y]$ and \equiv is a congruence on the median set X .

Case (ii) The cell $[a, c]$ has three elements, say $[a, c] = \{a, x, c\}$. Assuming $a \sim x$ it follows by the assumption (3) that $c \preceq a$, therefore $c \preceq x$. On the other hand, $x \preceq c$ since $c \in [x, b]$ and $x \preceq b$. Thus $x \sim c$, contrary to $x \equiv b, c \in [x, b], c \neq x$. Consequently, $a \not\sim x$, therefore $a \equiv x$ since $d(a, x) = 1$, and hence $a \equiv b$ as required.

(2) \implies (1). To conclude that the binary operation \bullet induced by the preorder \preceq is a quasidirection on X , by [2, Proposition 3.7.] it suffices to show that $a \bullet b \equiv a$ for all $a, b \in X$. Thanks to the assumption (2) we have to check the identity $a \bullet (a \bullet b) = a \bullet b$, i.e. $U_{a, a \bullet b} = \{a \bullet b\}$. Obviously, $a \bullet b \in U_{a, a \bullet b}$ since $a \preceq a \bullet b$. On the other hand, for any $c \in U_{a, a \bullet b}$ we get $a \preceq c$ and $b \preceq a \bullet b \preceq c$, and hence $c \in U_{a,b}$. Since $c \in U_{a, a \bullet b} \subseteq [a, a \bullet b]$, and $a \bullet b = \bigvee_a U_{a,b} \in [a, c]$, it follows that $c = a \bullet b$ as required. \square

Remark 2.5. Consider the simplicial tree X with three vertices a, b, c and two geometric edges (a, c) and (c, b) . Let \preceq be the complement in $X \times X$ of the subset $\{(a, b), (a, c)\}$. The relation \preceq is a preorder on X which is compatible with the tree structure, and any pair of elements of X is bounded above with respect to \preceq . However the preorder \preceq does not induce a quasidirection on X since the condition (3) above is not satisfied. Indeed, $c \preceq b$ but $a \not\preceq b$. Notice that in this simple case, $Qdir(X) \cong Fold(X)$ is naturally identified with the directed median set

$$\{\{a\}, \{b\}, \{c\}, [a, c] = \{a, c\}, [b, c] = \{b, c\}, X = [a, b] = \{a, b, c\}\}$$

of cardinality 6, consisting of the nonempty convex subsets of X .

3 Some classes of median groups

As shown in [9], the class of partially commutative Artin-Coxeter groups is naturally embedded into a larger class of median groups consisting of the so called \perp -groups, as defined in Introduction. Since the right-angled Artin groups form a proper subclass of the partially commutative Artin-Coxeter groups, it is natural to look for a proper subclass of the \perp -groups which is adequate for the investigation of the arboreal structure of right-angled Artin groups.

First of all notice that any l -group $(G, \cdot, \leq, \wedge, \vee)$, not necessarily commutative, has a canonical structure of median group. Indeed, as the underlying lattice of G is distributive, G has a canonical structure of median set with the median defined by $Y(x, y, z) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x) = (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$. Obviously, the median

operation is compatible with the multiplication, so G becomes a median group. Notice that $x \subset y$ iff $x_+ \leq y_+$ and $x_- \leq y_-$, $(x \cap y)_+ = x_+ \wedge y_+$, $(x \cap y)_- = x_- \wedge y_-$, where $x_+ = x \vee 1$, $x_- = (x^{-1})_+ = (x \wedge 1)^{-1}$.

For $x, y \in G$, $x \perp y$ iff x and y are orthogonal (or disjoint) as elements of the l -group G , i.e. $|x| \wedge |y| = 1$, where $|x| = x \vee x^{-1} = x_+ x_-$. Consequently, G is a \perp -group by [16, Proposition 3.1.3.] Notice that the \perp -group G above is simplicial iff it is Abelian, freely generated by the minimal positive elements.

Moreover, in a l -group G , the following are satisfied: $x \subset y \implies x^{-1} \subset y^{-1}$, $x \cap y = 1 \implies xz \cap yz \subset z$, and $x \cup x^{-1} \neq \infty \implies x = 1$.

Inspired by the properties above satisfied by l -groups, we introduce the following classes of median groups :

Definition 3.1. A median group G is said to be an A_i -group, $i = 1, 2, 3, 4$, if G satisfies the corresponding condition

- (A₁) $x \cup y \neq \infty$ and $x^{-1} \subset y^{-1} \implies x \subset y$
- (A₂) $x \cap y = x^{-1} \cap z = y^{-1} \cap z = 1 \implies xz \cap yz \subset z$
- (A₃) $x \cup x^{-1} \neq \infty \implies x^2 = 1$
- (A₄) $x \cup x^{-1} \neq \infty \implies x = 1$

The median group G is said to be an A -group if G is a \perp -group and also an A_i -group for $i = 1, 2, 4$.

Notice that the class of A_i -groups, $i = 1, 2, 3, 4$, as well as the class of A -groups, is closed under arbitrary products.

Remarks 3.2. (1) The l -groups are A -groups.

(2) Obviously, the locally linear median groups are \perp -groups. They are also A_2 -groups. Indeed, assume that G is a locally linear median group, and $x, y, z \in G$ satisfy the identities $x \cap y = x^{-1} \cap z = y^{-1} \cap z = 1$. Setting $u := xz \cap yz$, we have by assumption $x, u \in [1, xz]$ and $y, u \in [1, yz]$. As G is locally linear, we distinguish the following four cases.

- (i) : $u \subset x, u \subset y$. Then $u \subset x \cap y = 1$, and hence $u = 1 \subset z$.
- (ii) : $x \subset u, y \subset u$. Then either $x \subset y$ or $y \subset x$, therefore either $x = 1$ or $y = 1$ since $x \cap y = 1$ by assumption. Consequently, $u \subset z$.
- (iii) : $x \subset u, u \subset y$. Thus $x \subset y$, therefore $x = x \cap x \subset x \cap y = 1$, so $x = 1$, and hence $u \subset z$.
- (iv) : $u \subset x, y \subset u$. As in (iii), we get $u \subset z$ as desired.

On the other hand, the locally linear A_1 -groups are obviously A_3 -groups, but they are not necessarily A_4 -groups; for instance, the cyclic group of order 2 satisfies (A_i) , $i = 1, 2, 3$, while (A_4) is not satisfied.

(3) The locally linear median groups, and hence the \perp -groups too, are not necessarily A_1 -groups. To provide an example of a locally linear median group which is not an A_1 -group, we define a semilattice operation \cap on the set \mathbb{Z} of integers as follows :

$$n \cap m = \begin{cases} \min(n, m) & \text{if } n, m \text{ are even } \geq 0 \\ \max(n, m) & \text{if } n, m \text{ are either even } \leq 0 \text{ or odd} \\ n & \text{if } n \geq 0 \text{ is even, and } m \text{ is odd} \\ m & \text{if } m \geq 0 \text{ is even, and } n \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

It follows that $n \subset m$ iff one of the following four conditions is satisfied :

- (i) $0 \leq n \leq m$ and n, m are even
- (ii) $m \leq n \leq 0$ and n, m are even
- (iii) $m \leq n$ and n, m are odd
- (iv) $n \geq 0$ is even and m is odd.

One checks that \mathbb{Z} with the usual addition and the operation \cap as defined above becomes a locally linear median group G which is not an A_1 -group since $m \cup n = m \neq \infty, -m \subset -n$, and $m \not\subset n$ whenever m and n are odd integers such that $m < n$. Notice that the cell $[0, m] = \{n \in \mathbb{Z} \mid n \subset m\}$ is finite for m even : $[0, m] = \{n \in 2\mathbb{Z} \mid 0 \leq n \leq m\}$ for $m \geq 0$, resp. $[0, m] = \{n \in 2\mathbb{Z} \mid m \leq n \leq 0\}$ for $m \leq 0$, while $[0, m] = \{n \in 2\mathbb{Z} \mid n \geq 0\} \cup \{n \in 2\mathbb{Z} + 1 \mid n \geq m\}$ is infinite for m odd. Since $2 \cap -2 = 0$ and $2n \subset 1$ for all $n \geq 0$, it follows that G is not Archimedean (cf. Definition 3.3.) By contrast, the Archimedean \perp -groups are A_1 -groups (see Proposition 3.4.)

(4) For a locally linear median group G , the following assertions are equivalent.

- (i) G is an A -group.
- (ii) G is an A_i -group, $i = 1, 4$.
- (iii) $\forall x \in G \setminus \{1\}, x \not\subset x^{-1}$, and $\forall x \in G \setminus \{1\}, y \in G, x \subset xy \implies xy \not\subset y$.
- (iv) $\forall x \in G \setminus \{1\}, y, z \in G, [xy, xz] \not\subset [y, z]$.

(i) \iff (ii) follows by (2), (ii) \implies (iii) holds in all median groups, while (iii) \implies (ii) holds in locally linear median groups. On the other hand, (iii) \iff (iv) in locally linear median groups by [6, Lemma 3.2.]

(5) (A_1) and (A_2) do not imply (\perp) , resp. (A_3) . For instance, let $G = \mathbb{Z}/4\mathbb{Z}$ be the cyclic group of order 4. The canonical order \subset on $(G, S = \{1 \bmod 4\})$ makes G a simplicial locally boolean A_i -group for $i = 1, 2$. However G is not a \perp -group since $1 \perp -1$, while $1 \cup -1 = 2 \neq 0 = 1 + (-1)$. Moreover G is not an A_3 -group since $1 \cup -1 = 2 \neq \infty$ but $1 + 1 = 2 \neq 0$. By contrast, the conditions $(A_i), i = 1, 2, 3$, are obviously satisfied by any locally boolean \perp -group G since, according to [10, Corollary 2.1.], G is isomorphic to a subdirect product of a power set $(\mathbb{Z}/2\mathbb{Z})^I$ with the canonical group and median operations.

(6) There exist A_4 -groups (and hence A_3 -groups too) which are not \perp -groups and A_i -groups for $i = 1, 2$. Indeed, given an ordered field (K, \leq) , let $G = K_{>0}$ denote the multiplicative group of positive elements of K , with the action $G \times K \longrightarrow K, (x, a) \mapsto xa$ of G on the additive group K .

The total order on K makes G and K locally linear A_i -groups for $i = 1, 4$, and G acts as a group of automorphisms of the median group K . The semidirect product $H := K \rtimes G$, with $(a, x)(b, y) := (a + xb, xy), (a, x) \cap (b, y) := (a \cap b, x \cap y)$, for $a, b \in K, x, y \in G$, is an A_4 -group but it is not a \perp -group and an A_i -group for $i = 1, 2$. Indeed, assuming $(a, x) \in H$ such that $(a, x) \cup (a, x)^{-1} = (a, x) \cup (-x^{-1}a, x^{-1}) \neq \infty$ it follows that $x \cup x^{-1} \neq \infty$, therefore $x = 1$, and $a \cup -a \neq \infty$, and hence $a = 0$. Thus $(a, x) = (0, 1)$ is the neutral element of H , therefore H is an A_4 -group. To check that H is not a \perp -group, let $a \in K, x \in G$ be such that $a \neq 0, x \neq 1$. Obviously, $(a, 1) \cap (0, x) = (0, 1)$ and $(a, 1) \cup (0, x) = (a, x) = (a, 1)(0, x) \neq (0, x)(a, 1) = (xa, x)$, so H is not a \perp -group. To verify that H is not an A_1 -group, let $a \in K$ be such that $0 < a < 1$, and set $x := (1, 1), y := (a, a)$. We get $x \cup y = (1, a) \neq y$, though $x^{-1} = (-1, 1) \subset (-1, a^{-1}) = y^{-1}$. Finally, to check that H is not an A_2 -group, set

$x := (2, 2^{-1}), y := (0, 2), z := (1, 1)$. We obtain $x \cap y = x^{-1} \cap z = y^{-1} \cap z = (0, 1)$ but $xz \cap yz = (2 + 2^{-1}, 2^{-1}) \cap (2, 2) = (2, 1) \not\subset z$.

As we have seen in Remarks 2.3. (5), (A_2) does not imply (\perp) , however the converse is still open :

Question. Does the condition (\perp) implies (A_2) ?

Partial answers to the question above are provided by Remarks 2.3. (1), (2), (5), and Corollary 3.5.

Definition 3.3. A \perp -group G is called Archimedean if for every $x \in G$ satisfying $x \cap x^{-1} = 1$, i.e. $x \subset x^2$, and for every $y \in G$, there exists $n \geq 0$ such that $x^n \cap y = x^m \cap y$ for all $m \geq n$.

The Archimedean totally ordered groups, identified by Hölder's theorem with subgroups of the additive ordered group $(\mathbb{R}, +)$ of reals, and the simplicial \perp -groups, i.e. the partially commutative Artin-Coxeter groups, are natural examples of Archimedean \perp -groups.

Proposition 3.4. Any Archimedean \perp -group is an A_i -group for $i = 1, 3$.

Proof. Given an archimedean \perp -group G , let $x, y \in G$ be such that $x \cup y \neq \infty$ and $x^{-1} \subset y^{-1}$. To conclude that G is an A_1 -group we have to show that $x \subset y$. Setting $z := x \cap y, u := x^{-1}z, v := y^{-1}z$, it follows that $u^{-1} \perp v^{-1}$, therefore $u \perp v$ by [9, Lemma 2.2.4.]

On the other hand, $u \subset x^{-1} \subset y^{-1}$ and $v \subset y^{-1}$ imply $u \cup v = u \bullet v = v \bullet u \subset y^{-1} = v \bullet z^{-1}$, therefore $u \subset z^{-1}$ and $u \cap u^{-1} \subset z^{-1} \cap u^{-1} = 1$, in particular $u^n \subset u^{n+1}$ for all $n \geq 0$ by [9, Lemma 2.2.3.] It remains to show by induction that $u^n \subset z^{-1}$ for all $n \geq 0$ to conclude thanks to the archimedeanity of G that $u = 1$, i.e. $x = z \subset y$ as desired. Assuming $u^n \subset z^{-1} = u^n \bullet z'$ for some $n \geq 0$, we get $u^n \bullet u \bullet z' = u \bullet z^{-1} = x^{-1} \subset y^{-1} = v \bullet z^{-1} = v \bullet \underbrace{u \dots u}_{n \text{ factors}} \bullet z' = u^n \bullet v \bullet z'$, therefore

$u \subset z'$ and hence $u^{n+1} \subset z^{-1}$ as required.

To check that G is an A_3 -group, let $x \in G$ be such that $x \cup x^{-1} \neq \infty$, and let $y = x \cap x^{-1}$. Setting $u := x^{-1}y, v := xy$, we obtain $u^{-1} \perp v^{-1}$, therefore $u \perp v$ and $u \cup v = u \bullet v = v \bullet u$ by [9, Lemma 2.2.4.] Thanks to the archimedeanity of G it suffices to show that $(uv)^n \subset (uv)^{n+1} \subset y$ for all $n \geq 0$ to conclude that $v \perp u = v^{-1}$, so $u = v = 1$, and hence $x^2 = 1$ as desired. Since $x = y \bullet u^{-1} = v \bullet y^{-1}$ it follows by [10, Lemma 2.1.] and [9, Lemma 2.2.4.] that $u \subset y$ and $v \subset y$, and hence $u \cup v = u \bullet v = v \bullet u \subset y$. Setting $y' := (uv)^{-1}y$, we get further $u \bullet y' = y'^{-1} \bullet v^{-1}$, therefore, again by [10, Lemma 2.1.] and [9, Lemma 2.2.4.], $u \bullet v \subset y'^{-1}$. Setting $y'' := y'uv$, we obtain $y'' \bullet u^{-1} = v \bullet y''^{-1}$ and hence as above $u \bullet v \subset y''$, therefore $(uv)^2 = u \bullet v \bullet u \bullet v \subset y$. Thus by repeatedly applying the procedure above we obtain $(uv)^n \subset (uv)^{n+1} \subset y$ for all $n \geq 0$ as required. \square

Corollary 3.5. Any simplicial \perp -group is an A_i -group for $i = 1, 2, 3$.

Proof. The cases $i = 1, 3$ are immediate by Proposition 3.4. since the simplicial \perp -groups are Archimedean. To prove the case $i = 2$, assume that G is a simplicial

\perp -group, and let $x, y, z \in G$ be such that $x \cap y = x^{-1} \cap z = y^{-1} \cap z = 1$. To show that $u := (x \bullet z) \cap (y \bullet z) \subset z$, we argue by induction on the length $d := l(u)$ of u over the generating set $\tilde{S} = \{s \in G \setminus \{1\} \mid [1, s] = \{1, s\}\}$ of G . The case $d = 0$ is trivial, so let us assume $d \geq 1$, say $u = s \bullet v$ with $s \in \tilde{S}$. We distinguish the following three possibilities :

(i) : $s \subset x$, say $x = s \bullet x'$. As $s \subset y \bullet z$ and $s \cap y \subset x \cap y = 1$, we obtain $s \perp y$, and hence $s \subset z$, say $z = s \bullet z'$, and $y \bullet s = s \bullet y$. Simplifying with s , it follows that $v = x' \bullet s \bullet z' \cap y \bullet z'$. Notice that $x' \bullet s \cap y = 1$. Indeed, assuming the contrary, there exists $t \in \tilde{S}$ such that $t \subset x' \bullet s \cap y$, and hence $t \subset x' \cap y$ and $s \perp t$ since $s \perp y$. Consequently, $t \subset x = s \bullet x'$, therefore $t \subset x \cap y = 1$, a contradiction. Since $l(v) = d - 1$ it follows by the induction hypothesis that $v \subset z'$, and hence $u = s \bullet v \subset s \bullet z' = z$.

(ii) : $s \subset y$. We proceed as in the case (i).

(iii) : $s \cap x = s \cap y = 1$. It follows that $s \perp x, s \perp y$ and $s \subset z$, say $z = s \bullet z'$, therefore $v = x \bullet z' \cap y \bullet z'$. As $x \cap y = 1$ and $l(v) = d - 1$, the induction hypothesis implies $v \subset z'$ and hence $u \subset z$. \square

Corollary 3.6. *The necessary and sufficient condition for a simplicial \perp -group to be an A -group is that it is an A_4 -group.*

As an immediate consequence of [9, Theorem 2.4.1.], we obtain the following characterization of right-angled Artin groups.

Corollary 3.7. *Given a group G with a set $S \subseteq G$ of generators, let \subset denote the partial order on G induced by the canonical length function on (G, S) . Then, the following assertions are equivalent.*

- (1) $1 \notin S, S \cap S^{-1} = \{s \in S \mid s^2 = 1\}$, and the order \subset makes G an A -group.
- (2) $S \cap S^{-1} = \emptyset$, and the order \subset makes G a \perp -group.
- (3) (G, S) is a right-angled Artin group, i.e. G admits the presentation

$$G \cong \langle S; [s, t] = 1 \text{ for } s, t \in S, s \neq t, st = ts \text{ holds in } G \rangle$$

Thus the right-angled Artin groups are identified with the simplicial A -groups.

We end the present section with four useful lemmas.

Lemma 3.8. *Let G be an A_1 -group, and let $x, y \in G$ be such that $xy = yx$. Then, $x^{-1} \cap y = 1 \iff y^{-1} \cap x = 1$.*

Proof. Let $x, y \in G$ be such that $xy = yx$ and $x^{-1} \cap y = 1$, i.e. $xy = x \bullet y$. Setting $z = y^{-1} \cap x$, we get $yz = y \cap yx \subset yx = xy$, and $xyz = Y(x, xy, yx) \subset xy$ since $x \cap yx \subset x \subset xy$, therefore $yz \cup xyz \neq \infty$. As $x^{-1} \cap yz \subset x^{-1} \cap y = 1$, we get $(yz)^{-1} \subset (yz)^{-1}x^{-1} = (xyz)^{-1}$, and hence $yz \subset xyz \subset xy = yx$ by (A_1) . Consequently, $z \subset x^{-1}z \subset x^{-1}$. By symmetry, it follows that $z \subset y$ too, therefore $z \subset x^{-1} \cap y = 1$, i.e. $y^{-1} \cap x = 1$, as desired. \square

Lemma 3.9. *Assume that G is a \perp -group satisfying (A_1) and (A_2) , and let $x, y, z \in G$ be such that $xyz = x \bullet y \bullet z = z \bullet y \bullet x$ and $x \cap z = 1$. Then, $xy = yx, xz = zx$, and $yz = zy$.*

Proof. First note that $x \perp z$, and hence $xz = zx$ since G is a \perp -group by assumption. Consequently, $u := x^{-1}yx = z^{-1}yz$, so we have to show that $u = y$. As G is a \perp -group, $x \perp z$ implies $y \bullet x = x \bullet u$ and $y \bullet z = z \bullet u$, and hence $y = y \bullet x \cap y \bullet z = x \bullet u \cap z \bullet u \subset u$ by (A_2) . On the other hand, $x \perp z$ implies $x^{-1} \perp z^{-1}$ by [9, Lemma 2.2.4.], therefore $u^{-1} = u^{-1} \bullet x^{-1} \cap u^{-1} \bullet z^{-1} = x^{-1} \bullet y^{-1} \cap z^{-1} \bullet y^{-1} \subset y^{-1}$ by (A_2) again. As $y \subset u$ and $u^{-1} \subset y^{-1}$, it follows by (A_1) that $u = y$ as required. \square

Lemma 3.10. *Assume that G is a \perp -group satisfying (A_1) and (A_2) . Then, for all $x \in G$, $[1, x] \cap Z_G(x) = \{y \in G \mid y \subset x, xy = yx\}$ is a sublattice of the cell $[1, x]$.*

Proof. Let $y, z \in [1, x] \cap Z_G(x)$, $u := y \cap z$, $y' := u^{-1}y$ and $z' := u^{-1}z$. As $y' \subset u^{-1}x$, $z' \subset u^{-1}x$ and $y' \cap z' = 1$, we get $y' \perp z'$, therefore $y \cup z = y \vee z = u \bullet y' \bullet z' = u \bullet z' \bullet y'$ since G is a \perp -group. Set $v := (y \cup z)^{-1}x$. Since y and z belong to $Z_G(x)$, and G satisfies (A_1) , it follows by Lemma 3.8. that $z' \bullet v \bullet u \bullet y' = u \bullet y' \bullet z' \bullet v = x = u \bullet z' \bullet y' \bullet v = y' \bullet v \bullet u \bullet z'$. According to Lemma 3.9. we get $z'vu = vuz'$ and $y'vu = vuy'$, therefore $x = uy'z'v = z'vuy' = z'y'vu = y'z'vu = y'vuz' = vuy'z'$, and hence $y \cap z$ and $y \cup z$ belong to $Z_G(x)$ as desired. \square

Lemma 3.11. *Let G be an Abelian median group. Then, the following assertions are equivalent.*

- (1) G is an A -group.
- (2) G is an A_1 -group, and $x \cap x^{-1} = 1$, i.e. $x \subset x^2$, for all $x \in G$.

Proof. (1) \implies (2) : We have only to show that $x \cap x^{-1} = 1$ for all $x \in G$. Let $x \in G$, and set $u := x \cap x^{-1}$. As G is Abelian and satisfies (A_1) , it follows by Lemma 3.8. that $u, u^{-1} \subset x$, therefore $u \cup u^{-1} \neq \infty$, and hence $u = 1$ by (A_4) .

(2) \implies (1) : We have to show that G is a \perp -group satisfying (A_2) and (A_4) .

Assuming that $x \perp y$, let us show that $u := x \cup y = xy$. As $x, y \subset u$ and G is an Abelian A_1 -group, it follows by Lemma 3.8. that $x^{-1}, y^{-1} \subset u^{-1}$, therefore $x, y \subset xy = yx$ since $x^{-1} \cap y, y^{-1} \cap x \subset u \cap u^{-1}$ and $u \cap u^{-1} = 1$ by assumption. Consequently, $u = Y(x, y, xy) = xy(x^{-1} \cap y^{-1}) = xy$ since $x \perp y \implies x \cap y = 1 \implies x^{-1} \cap y^{-1} = 1$ again by Lemma 3.8. Thus G is a \perp -group.

To show that G satisfies (A_2) , let $x, y, z \in G$ be such that $x \cap y = x^{-1} \cap z = y^{-1} \cap z = 1$. As G is an Abelian A_1 -group, we deduce by Lemma 3.8. that $z \subset xz = zx$ and $z \subset yz = zy$, and hence $xz \cap yz = z(x \cap y) = z \subset z$ as desired.

Finally, to show that G satisfies (A_4) , let $x \in G$ be such that $u := x \cup x^{-1} \neq \infty$. As G is a \perp -group, we get $u = x(x \cap x^{-1})^{-1}x^{-1}$, and hence $u = 1$ as required, since G is Abelian and $x \cap x^{-1} = 1$ by assumption. \square

Corollary 3.12. *Let G be an Abelian locally linear median group. Then, the following assertions are equivalent.*

- (1) G is an A -group.
- (2) $x \cap x^{-1} = 1$ for all $x \in G$.
- (3) There exist only two opposite total orders on G making G a totally ordered Abelian group whose associated median group is the given locally linear median group G .

Proof. (1) \iff (2) : By Lemma 3.11., we have to show that (2) \implies (A₁) in Abelian locally linear median groups. Let $x, y \in G$ be such that $x \cup y \neq \infty$ and $x^{-1} \subset y^{-1}$. By locally linearity, $x \cup y \neq \infty \implies$ either $x \subset y$ or $y \subset x$. In the former case we are done, while in the latter case, $y \subset x \iff x^{-1}y \subset x^{-1} \implies x^{-1}y \subset y^{-1} \implies x^{-1}y \cap y \subset y^{-1} \cap y = 1 \iff y^{-1} \subset y^{-1}x^{-1}y = x^{-1}$, therefore $x = y \subset y$ as desired.

For (2) \iff (3) see [6, Corollary 3.4., Remark] \square

4 Cyclically reduced elements in A -groups

In the rest of this Section, as well as in Sections 5 and 6, G will denote an arbitrary A -group.

The basic notion of a cyclically reduced word in a free group extends naturally to A -groups as follows.

Definition 4.1. *An element $w \in G$ is said to be cyclically reduced if $w \cap w^{-1} = 1$.*

Among the A -groups for which all elements are cyclically reduced, we mention the l -groups and the Abelian A -groups (by Lemma 3.11.)

Lemma 4.2. *Given $w \in G$, let $u := w \cap w^{-1}$ and $v := u^{-1}wu$. Then, u is the unique element of G for which v is cyclically reduced and $w = u \bullet v \bullet u^{-1}$.*

Proof. By definition of u we obtain $w = u \bullet (u^{-1}w) = (wu) \bullet u^{-1}$, in particular $wu \subset w$. To show that $u \subset wu$ set $u' := u \cap wu$. It follows that $(u'^{-1}u) \bullet (u^{-1}w) = (u'^{-1}wu) \bullet u^{-1}$, therefore $u'^{-1}u \subset u^{-1}$ since G is a \perp -group and $u'^{-1}u \cap u'^{-1}wu = 1$. As, on the other hand, $(u'^{-1}u)^{-1} = u^{-1}u' \subset u^{-1}$ since $u' \subset u$, we get $(u'^{-1}u) \cup (u'^{-1}u)^{-1} \neq \infty$, and hence $u = u' \subset wu$ by (A₄). To show that v is cyclically reduced, set $v' := v \cap v^{-1}$ and $w' := v'^{-1}vv'$. With the argument above we obtain $w = u \bullet v \bullet u^{-1} = u \bullet v' \bullet w' \bullet v'^{-1} \bullet u^{-1}$, therefore $u \subset u \bullet v' \subset w \cap w^{-1} = u$, i.e. $v' = 1$ as required.

To prove the uniqueness part of the statement, let $s \in G$ be such that $s \subset ws \subset w$ and $t := s^{-1}ws$ is cyclically reduced. As $s \subset w \cap w^{-1}$, it remains to check that $ws \subset w^2$ to conclude that $s = w \cap w^{-1}$. As $s^{-1}w \cap s^{-1}w^{-1} = t \bullet s^{-1} \cap t^{-1} \bullet s^{-1}$ and $t \cap t^{-1} = 1$, it follows by (A₂) that $s^{-1}w \cap s^{-1}w^{-1} \subset s^{-1} \cap s^{-1}w = 1$, and hence $ws \subseteq w^2$ as desired. \square

Lemma 4.3. *For all $w \in G$ and for all natural numbers $n, m \geq 1$, $w^n \cap w^{-m} = w \cap w^{-1}$.*

Proof. Let $u := w \cap w^{-1}$ and $v := u^{-1}wu$. By Lemma 4.2., $v \cap v^{-1} = 1$, therefore $v^n \cap v^{-m} = 1$ for $n, m \in \mathbb{N}$ according to [9, Lemma 2.2.3.], in particular, v^n is cyclically reduced for all $n \in \mathbb{Z}$. To conclude that $w^n \cap w^{-m} = u$ for all $n \geq 1, m \geq 1$, it suffices to show that $w^n = u \bullet v^n \bullet u^{-1}$ for all $n \geq 1$. Indeed the last condition implies the identity $w^n \cap w^{-m} = u \bullet (v^n \bullet u^{-1} \cap v^{-m} \bullet u^{-1}) = u$ since $t := v^n \bullet u^{-1} \cap v^{-m} \bullet u^{-1} \subset u^{-1}$ by (A₂), whence $t = t \cap u^{-1} \subset v^n \bullet u^{-1} \cap u^{-1} = 1$. To check that $w^n = u \bullet v^n \bullet u^{-1}$ for $n \geq 1$, we argue by induction on n . The case $n = 1$ is assured by Lemma 4.2., so assuming $w^n = u \bullet v^n \bullet u^{-1}$ for some $n \geq 1$, we have to show that $w^{n+1} = u \bullet v^{n+1} \bullet u^{-1}$.

As $u^{-1} \cap v^n = 1$ by the induction hypothesis and G is a \perp -group, it follows that $u^{-1} \cap v^{n+1} = u^{-1} \cap v^n \bullet v \subset u^{-1} \cap v = 1$, i.e. $u \subset uv^{n+1}$. Thus it remains to show that $s := v^{-n-1} \bullet u^{-1} \cap u^{-1} = 1$. As $s \subset v^{-n-1} \bullet u^{-1}, v^{-1} \subset v^{-n-1} \bullet u^{-1}$ and $s \cap v^{-1} \subset u^{-1} \cap v^{-1} = 1$, it follows that $s \perp v^{-1}$, therefore $s \perp v$ by [9, Lemma 2.2.4.] Consequently, $v \bullet u^{-1} = v \bullet s \bullet (s^{-1}u^{-1}) = s \bullet v \bullet (s^{-1}u^{-1})$, and hence $s \subset u^{-1} \cap v \bullet u^{-1} = 1$, i.e. $s = 1$ as required. \square

Corollary 4.4. *For $w \in G$ and $0 \neq n \in \mathbb{Z}$, w^n is cyclically reduced if and only if w is cyclically reduced.*

Corollary 4.5. *The A -groups are torsion-free.*

Proof. Let $w \in G$ and $n \geq 1$ be such that $w^n = 1$. By Lemma 4.3. we get $w \cap w^{-1} = w^n \cap w^{-1} = 1$, and hence $w \subset w^n = 1$ by [9, Lemma 2.2.3.], so $w = 1$. \square

Remark 4.6. Let (G, S) be a right-angled Artin group. By Corollaries 3.7. and 4.5., G is torsion-free. For $w \in G, n \geq 1$, it follows by Lemmas 4.2. and 4.3. that $l(w^n) = 2l(w \cap w^{-1}) + nl((w \cap w^{-1})^{-1}w(w \cap w^{-1}))$. Given $w \in G$ and $s \in \tilde{S} = S \cup S^{-1}$, set $u := w \cap w^{-1}, v := u^{-1}wu, w' := sws^{-1}, u' := w' \cap w'^{-1}, v' := u'^{-1}w'u'$. We distinguish the following four cases:

Case (1) $sw = s \bullet w$ and $ws^{-1} = w \bullet s^{-1}$: Then, by the condition (F') [9, 1.5.], either $w' = s \bullet w \bullet s^{-1}$ in which case $u' = s \bullet u, v' = v$, and $l(w') = l(w) + 2$ or $sw = ws$, i.e. $w' = w$.

Case (2) $s^{-1} \subset u = w \cap w^{-1}$: Then, $u' = su, v' = v$, and $l(w') = l(w) - 2$.

Case (3) $s^{-1} \subset w$ and $ws^{-1} = w \bullet s^{-1}$: As $s^{-1} \subset w = u \bullet v \bullet u^{-1}$ and $s^{-1} \cap u \subset s^{-1} \cap w^{-1} = 1$, it follows that $s^{-1} \perp u$, therefore $s^{-1} \perp u^{-1}$ by [9, Lemma 2.2.4.], and hence $v = s^{-1} \bullet (sv)$ by (\perp) . Consequently $u' = sus^{-1} = u, v' = (sv) \bullet s^{-1}$, and $l(w') = l(w)$.

Case (4) $s^{-1} \subset w^{-1}$ and $sw = s \bullet w$: Applying Case (3) to w^{-1} we obtain $v = (vs^{-1}) \bullet s, u' = sus^{-1} = u, v' = s \bullet (vs^{-1})$, and $l(w') = l(w)$.

The discussion above implies that for given cyclically reduced elements $w, w' \neq 1$ in (G, S) , the necessary and sufficient condition for w and w' to be conjugate is that $l(w) = l(w')$ and there exist a sequence $w_1 = w, w_2, \dots, w_n = w'$ of length $n \leq l(w)!$ and $s_i \in \tilde{S}$ such that $s_i \subset w_i$ and $w_{i+1} = (s_i^{-1}w_i) \bullet s_i$ for $i < n$. Consequently, as the word problem on (G, S) is solvable [24], [9, 1.5.], it follows that the conjugacy problem on (G, S) is solvable too.

We end this section with four technical lemmas concerning A -groups which will be useful later.

Lemma 4.7. *Given the elements x, w of the A -group G , set $u := w \cap w^{-1}, v := u^{-1}wu, w' := xwx^{-1}, u' := w' \cap w'^{-1}, v' := u'^{-1}w'u'$, and $z := x^{-1} \cap w^{-1}x^{-1}$. If $xw = x \bullet w$ then $xw^2 = x \bullet w^2, z \perp u, u' = (xz) \bullet u$, and $v' = z^{-1}vz$.*

In particular, if in addition w and w' are both cyclically reduced, then $z = x^{-1}$, i.e. $xw = w' \bullet x$.

Proof. We proceed step by step as follows:

1) $xw^2 = x \bullet w^2$: By Lemmas 4.2. and 4.3., $w = u \bullet v \bullet u^{-1}$ and $w^2 = u \bullet v \bullet v \bullet u^{-1}$. Setting $s := x^{-1} \cap w^2$, we get $s \cap u \bullet v \subset x^{-1} \cap w = 1$, whence $s \perp u \bullet v$ and $s \subset v \bullet u^{-1}$ by (\perp) . On the other hand, $s \perp u \bullet v$ implies $s \perp v$ and $s \perp u^{-1}$ by [9, Lemma 2.2.4.], therefore $s = s \cap v \bullet u^{-1} = 1$, as desired.

2) $z \perp u$, in particular, $z \bullet u = u \bullet z$: As $z \subset w^{-1}x^{-1}$ and $u \subset w^{-1} \subset w^{-1}x^{-1}$, it follows that $z \cup u \neq \infty$. Consequently, $z \perp u$ since $z \cap u \subset x^{-1} \cap w = 1$.

3) $xzu = (xz) \bullet u$: As $x^{-1} \cap u \subset x^{-1} \cap w = 1$, we get $z \subset x^{-1} \cap z \bullet u = x^{-1} \cap u \bullet z \subset z$ by 2) and (\perp) , therefore $z^{-1}x^{-1} \cap u = z^{-1}Y(x^{-1}, z, z \bullet u) = 1$, as required.

4) Set $a := x^{-1} \cap w^{-1} = x^{-1} \cap u \bullet v^{-1} \bullet u^{-1}$. As $a \subset z$ and $z \perp u$ by 2), it follows that $a \perp u$, and hence $a \subset v^{-1}$ by [9, Lemma 2.2.5.] Setting $x = y \bullet a^{-1}$ and $v = c \bullet a^{-1}$, we obtain

$$(4.1.) \quad y^{-1} \cap u \bullet c^{-1} \bullet u^{-1} = 1.$$

Thus $z = a \bullet b$, where $b := y^{-1} \cap u \bullet c^{-1} \bullet u^{-1} \bullet x^{-1} \subset x^{-1}$ and $b \perp u \bullet c^{-1} \bullet u^{-1}$ by (4.1.). As $z = a \bullet b \subset x^{-1}$ and $b \subset x^{-1}$, we get $b \subset a \bullet b$ by (A_1) , i.e. $z = a \bullet b = b \bullet a'$, with $a' = b^{-1}ab$. Setting $y = y' \bullet b^{-1}$, we obtain $x = y' \bullet b^{-1} \bullet a^{-1} = y' \bullet a'^{-1} \bullet b^{-1}$ and

$$(4.2.) \quad y'^{-1} \cap u \bullet c^{-1} \bullet u^{-1} \bullet a' \bullet y'^{-1} = 1.$$

5) It remains to show that $u' = xzu = xuz$, since then we get $v' = u'^{-1}u'u' = z^{-1}vz$, as required. As $u' = w' \cap w'^{-1} = xY(wx^{-1}, x^{-1}, w^{-1}x^{-1})$, we have to show that $Y(wx^{-1}, x^{-1}, w^{-1}x^{-1}) = z \bullet u = u \bullet z = z \cup u$. Since $x^{-1} \cap w^{-1}x^{-1} = z$ by definition, it remains to check that $wx^{-1} \cap x^{-1} = b \subset z$ and $wx^{-1} \cap w^{-1}x^{-1} = u \bullet b \subset u \bullet z$.

5.1.) We deduce from (4.1.) and (4.2.) that

$$wx^{-1} \cap x^{-1} = u \bullet c \bullet u^{-1} \bullet b \bullet y'^{-1} \cap b \bullet a' \bullet y'^{-1} = b \bullet (u \bullet c \bullet u^{-1} \bullet y'^{-1} \cap a' \bullet y'^{-1}) = b.$$

5.2.) It follows that $wx^{-1} \cap w^{-1}x^{-1} = u \bullet c \bullet u^{-1} \bullet b \bullet y'^{-1} \cap u \bullet a \bullet c^{-1} \bullet u^{-1} \bullet b \bullet a' \bullet y'^{-1} = u \bullet b \bullet t = b \bullet u \bullet t$, where $t = c \bullet u^{-1} \bullet y'^{-1} \cap a' \bullet c^{-1} \bullet u^{-1} \bullet a' \bullet y'^{-1}$, since $b \perp u \bullet c^{-1} \bullet u^{-1}$. By 1) we get $w^{-2}x^{-1} = w^{-2} \bullet x^{-1} = u \bullet v^{-2} \bullet u^{-1} \bullet x^{-1} = u \bullet a \bullet c^{-1} \bullet a \bullet c^{-1} \bullet u^{-1} \bullet b \bullet a' \bullet y'^{-1} = u \bullet a \bullet b \bullet c^{-1} \bullet a' \bullet u^{-1} \bullet a' \bullet y'^{-1} = z \bullet u \bullet c^{-1} \bullet a' \bullet c^{-1} \bullet u^{-1} \bullet a' \bullet y'^{-1}$, therefore $c \bullet u^{-1} \cap a' \bullet c^{-1} \bullet u^{-1} \bullet a' \bullet y'^{-1} = 1$, and hence $t \subset y'^{-1} \cap u^{-1} \bullet y'^{-1}$. As $u^{-1} \cap t = 1$, it follows that $t \perp u^{\pm 1}$, in particular, $t \bullet u = u \bullet t$. On the other hand, since $z = b \bullet a' \perp u$, it follows by [9, Lemma 2.2.5.] that $a' \perp u^{\pm 1}$, and hence we deduce from (4.2) that $c \bullet u^{-1} \bullet y'^{-1} \cap a' = 1$, whence $t \subset y'^{-1} \cap c^{-1} \bullet u^{-1} \bullet a' \bullet y'^{-1}$. As $t \bullet u = u \bullet t$, we get $t \subset y'^{-1} \cap u \bullet c^{-1} \bullet u^{-1} \bullet a' \bullet y'^{-1} = 1$ (by (4.2)), therefore $t = 1$ as desired. \square

Remark 4.8. With the notation above, we have $z^{-1} \bullet v = v' \bullet z^{-1}$, $v = c \bullet a^{-1}$ and $v' = a'^{-1} \bullet c$, where $a = x^{-1} \cap w^{-1}$, $a' = z^{-1}az = b^{-1}ab$, $b = wx^{-1} \cap x^{-1}$.

Lemma 4.9. For all $x, y, w \in G$, $x \subset wx$, $y \subset wx$ and $y \cap w = 1 \implies y \subset x$.

Proof. By Lemma 4.2., $x^{-1}wx = u \bullet v \bullet u^{-1}$, where $u = x^{-1}wx \cap x^{-1}w^{-1}x$ and $v = u^{-1}x^{-1}wxu$. Applying Lemma 4.7. to the reduced pair $(x, x^{-1}wx)$, we get $w = u' \bullet v' \bullet u'^{-1}$, where $u' = w \cap w^{-1} = (xz) \bullet u$, $z = x^{-1} \cap x^{-1}w^{-1} = x^{-1}(x \cap w^{-1})$, so $xz = x \cap w^{-1}$, $v' = u'^{-1}wu^{-1}$. Consequently, $xz \subset wxz \subset w$. As $y \subset wx = x \bullet (x^{-1}wx) = (wxz) \bullet z^{-1}$ and $y \cap wxz \subset y \cap w = 1$, it follows that $y \perp wxz$, in particular, $y \perp xz$ and $y \subset z^{-1}$. Since $x = (xz) \bullet z^{-1} = (xz) \bullet y \bullet (y^{-1}z^{-1}) = y \bullet (xz) \bullet (y^{-1}z^{-1})$, we get $y \subset x$ as desired. \square

Lemma 4.10. *For $x, y, w \in G$, $x \subset w$ and $x \perp y \implies x \subset ywy^{-1}$.*

Proof. Setting $z = y^{-1} \cap w$, $z \subset y^{-1}$ and $x \perp y$ imply by [9, Lemma 2.2.4.] $x \perp z$ and $x \perp yz$. As $x \subset w$ and $z \subset w$, we get $w = x \bullet z \bullet w' = z \bullet x \bullet w'$, and $yw = (yz) \bullet (x \bullet w') = x \bullet (yz) \bullet w'$. It follows that $(yw)^{-1} \cap y^{-1} = w'^{-1} \bullet (yz)^{-1} \bullet x^{-1} \cap y^{-1} = w'^{-1} \bullet (yz)^{-1} \cap y^{-1}$ since $x^{-1} \perp y^{-1}$ by [9, Lemma 2.2.4.] We conclude that $x \subset ywy^{-1}$ as required. \square

Lemma 4.11. *Given $x \in G$ and a cyclically reduced element $w \in G$, set $a := x \cap w$, $b := x \cap w^{-1}$, $y := a^{-1}b^{-1}x$, and $u := a^{-1}wb$. The necessary and sufficient condition for the conjugate $x^{-1}wx$ of w to be cyclically reduced is that $y = ay \cap by$ and $y \perp u$.*

Proof. As $w \cap w^{-1} = 1$, we obtain $a \perp b$, $x = a \bullet b \bullet y = b \bullet a \bullet y$, and $z := a \bullet y \cap b \bullet y \subset y$ by (A_2) . Setting $w = a \bullet w'$, $b \subset w^{-1} = w'^{-1} \bullet a^{-1}$ and $b \perp a^{-1}$ (by [9, Lemma 2.2.4.]) imply $b \subset w'^{-1}$, i.e. $w = a \bullet u \bullet b^{-1}$, and $a \bullet y \cap u^{-1} \bullet a^{-1} = b \bullet y \cap u \bullet b^{-1} = 1$, in particular, $z \cap u = 1$. We get $x \cap wx = a \bullet b \bullet y \cap a \bullet u \bullet a \bullet y = a \bullet s$, where $s := b \bullet y \cap u \bullet a \bullet y \subset y$ by (A_2) since $b \cap u \bullet a = 1$. By symmetry, we obtain $x \cap w^{-1}x = b \bullet t$, where $t := a \bullet y \cap u^{-1} \bullet b \bullet y \subset y$. On the other hand, $wx \cap w^{-1}x = a \bullet u \bullet a \bullet y \cap b \bullet u^{-1} \bullet b \bullet y \subset y$ by (A_2) again since $a \bullet u \bullet a \cap b \bullet u^{-1} \bullet b \subset w^2 \cap w^{-2} = 1$.

Assuming that $x^{-1}wx$ is cyclically reduced, it follows that $y^{-1}s, y^{-1}t \subset x^{-1}wx \cap x^{-1}w^{-1}x = 1$, therefore $y = s = t = z$, and $y \cap u = z \cap u = 1$. As $y = s \subset u \bullet a \bullet y$, we also get $y \perp u$.

Conversely, assuming $y = z$ and $y \perp u$, we get $wx \cap w^{-1}x = y \subset x$, $x \cap wx = a \bullet y$, and $x \cap w^{-1}x = b \bullet y$, and hence $Y(wx, x, w^{-1}x) = x \cap Y(wx, x, w^{-1}x) = (x \cap wx) \cup (x \cap w^{-1}x) = (ay)y^{-1}(by) = aby = x$, so $x \in [wx, w^{-1}x]$, i.e. $x^{-1}wx$ is cyclically reduced. \square

5 Preorders induced by elements of A -groups

Given an element w of the A -group G , let \preceq_w denote the binary relation defined by $x \preceq_w y$ iff $y \in [x, wy]$, i.e. $x^{-1}y \subset x^{-1}wy$. Notice that $zx \preceq_{zwz^{-1}} zy \iff x \preceq_w y$ for all $x, y, z \in G$.

Lemma 5.1. *The relation \preceq_w is a preorder.*

Proof. As the reflexivity is trivial, it remains to check the transitivity of \preceq_w . Using a convenient conjugation, it suffices to show that $x \preceq_w y$ whenever $x \preceq_w 1$ (i.e. $x \cap w = 1$) and $1 \preceq_w y$ (i.e. $y \subset wy$). By Lemma 4.9. we get $x \cap wy \subset y$, therefore $Y(x, y, wy) = y$, i.e. $x \preceq_w y$ as required. \square

Let \sim_w denote the equivalence relation induced by the preorder \preceq_w .

Lemma 5.2. *The necessary and sufficient condition for $x \sim_w y$ is that $[x, wy] = [y, wx]$, i.e. $y^{-1}x \perp y^{-1}wy$.*

Proof. The non-trivial implication to prove is $x \underset{w}{\sim} y \implies y^{-1}x \perp y^{-1}wy$. Without loss we may assume that $y = 1$, so we have to show that $w \subset wx$ provided $x \subset wx$ and $x \cap w = 1$. Applying Lemma 4.7. to the reduced pair $(x, x^{-1}wx)$, we get $w = (wxz) \bullet (xz)^{-1}$ and $wx = x \bullet (x^{-1}wx) = (wxz) \bullet z^{-1}$, where $z = x^{-1} \cap x^{-1}w^{-1} = x^{-1}(x \cap w^{-1})$. As $x \subset wx, wxz \subset wx$ and $x \cap wxz \subset x \cap w = 1$, it follows by (\perp) that $x \subset z^{-1}$, whence $x = z^{-1}$ by (A_1) since $z \subset x^{-1}$. Consequently, $wx = w \bullet x$ as desired. \square

Lemma 5.3. *If $x \underset{w}{\preceq} y$ and $z \in [x, y]$, then $x \underset{w}{\preceq} z$ and $z \underset{w}{\preceq} y$.*

Proof. We may assume without loss that $x = 1$, so we have to show that $y \subset wy$ and $x \subset y$ imply $x \subset wz$ and $y \in [z, wy]$. As $z \subset y \subset wy$, we get obviously $y \in [z, wy]$, so it remains to prove that $z \subset wz$. In other words, we have to show that for $x, y, w \in G$, the pair (x, ywy^{-1}) is reduced whenever the triple (x, y, w) is reduced, i.e. $x \subset xy \subset xyw$. Assuming that the triple (x, y, w) is reduced, let us first apply Lemma 4.7. to the reduced pair (y, w) . Setting $u = w \cap w^{-1}, v = u^{-1}wu, z = y^{-1} \cap w^{-1} \bullet y^{-1}, u' = w' \cap w'^{-1}, v' = u'^{-1}w'u'$, we obtain $z \perp u, u' = (yz) \bullet u$ and $v' \bullet z^{-1} = z^{-1} \bullet v$. Consequently, $1 = x^{-1} \cap y \bullet w = x^{-1} \cap (yz) \bullet z^{-1} \bullet u \bullet v \bullet u^{-1} = x^{-1} \cap u' \bullet v' \bullet u^{-1} \bullet z^{-1}$, therefore $s := x^{-1} \cap ywy^{-1} = x^{-1} \cap u' \bullet v' \bullet u'^{-1} = x^{-1} \cap u' \bullet v' \bullet u^{-1} \bullet (yz)^{-1} \subset (yz)^{-1}$ by (\perp) . As s and $yz \subset u'$ are bounded above by $u' \bullet v' \bullet u^{-1} \bullet (yz)^{-1}$, and $s \cap yx \subset x^{-1} \cap u' = 1$, we get $s \perp yz$, and hence $s \perp (yz)^{-1}$ by [9, Lemma 2.2.4.] Since, on the other hand, $s \subset (yz)^{-1}$, it follows that $s = 1$, i.e. the pair (x, ywy^{-1}) is reduced as desired. \square

The preorder $\underset{w}{\preceq}$ is compatible with the arboreal structure of G , as follows :

Proposition 5.4. *Given $x, y, a, b \in G, Y(a, b, y) \underset{w}{\preceq} Y(a, b, x)$ whenever $y \underset{w}{\preceq} x$. In particular, the equivalence relation $\underset{w}{\sim}$ is a congruence on the underlying median set of G .*

Proof. We may assume that $y = 1$, so we have to show that $x \subset wx \implies Y(a, b, x) \in [a \cap b, Y(wa, wb, wx)]$. Setting $c := Y(a, b, x) = (a \cap b) \cup (c \cap x)$, and $d := (a \cap b) \cap (c \cap x) = a \cap b \cap x$, we get $c = (a \cap b)d^{-1}(c \cap x) = (c \cap x)d^{-1}(a \cap b), d^{-1}x \subset d^{-1}wx$, and $d^{-1}(a \cap b) \perp d^{-1}(c \cap x)$, whence $(a \cap b)^{-1}d \perp d^{-1}(c \cap x)$ by [9, Lemma 2.2.4.] As $d^{-1}(c \cap x) \subset d^{-1}x$, it follows by Lemma 5.3. that $d^{-1}(c \cap x) \subset d^{-1}w(c \cap x)$, therefore $d^{-1}(c \cap x) \subset ((a \cap b)^{-1}d)(d^{-1}w(c \cap x))(d^{-1}(a \cap b)) = (a \cap b)^{-1}wc$, according to Lemma 4.10. Multiplying with $a \cap b$, we obtain $c \in [a \cap b, wc]$ as required. \square

For any $x \in G$, let \tilde{x}^w denote the $\underset{w}{\sim}$ -class of x . Note that \tilde{x}^w is a convex subset of G . For $x, y \in G$ such that $x \underset{w}{\preceq} y$, set $< x, y >_w := \{z \in G \mid x \underset{w}{\preceq} z \text{ and } z \underset{w}{\preceq} y\}$.

Corollary 5.5. *Let $x, y \in G$ be such that $x \underset{w}{\preceq} y$. The convex subset $< x, y >_w$ is the disjoint union $\bigsqcup_{z \in [x, y]} \tilde{z}^w$.*

Corollary 5.6. $\tilde{1}^w = \{x \in G \mid x \perp w\}$ is a convex subgroup of G .

Proof. The closure of $\tilde{1}^w$ under the operation $x \mapsto x^{-1}$ follows by [9, Lemma 2.2.4.] To check the closure of $\tilde{1}^w$ under multiplication, let $x, y \in \tilde{1}^w$, and set $z := x^{-1} \cap y$. Thus $xy = (xz) \bullet (z^{-1}y)$, where xz and $z^{-1}y$ belong to $\tilde{1}^w$ by [9, Lemma 2.2.5.], so we may assume from the beginning that $xy = x \bullet y$, with $x, y \in \tilde{1}^w$. As $x \perp w$ and $y \perp w$ we obtain $w \cap x \bullet y = w^{-1} \cap x \bullet y = 1$ by (\perp) , therefore $xy \perp w$ since $w \bullet x \bullet y = x \bullet w \bullet y = x \bullet y \bullet w$. \square

Lemma 5.7. *For all $n \geq 1$, the preorder \preceq_{w^n} induced by w^n equals \preceq_w .*

Proof. It suffices to show that $x \preceq_w 1 \iff x \preceq_{w^n} 1$, i.e. $x \cap w = 1 \iff x \cap w^n = 1$. It remains to argue as in the step 1) of the proof of Lemma 4.7. \square

6 Foldings induced by elements of A -groups

For any element w of an A -group G , let us define the mapping $\varphi_w : G \longrightarrow G$ by $\varphi_w(x) = Y(wx, x, w^{-1}x)$. The main goal of this section is to show that φ_w is a folding of the underlying median set of G . In the particular case of locally linear A -groups, this fact is a consequence of [6, Proposition 2.7.]

Notice that $\varphi_{xwx^{-1}}(y) = x\varphi_w(x^{-1}y)$ for all $x, y \in G$.

Lemma 6.1. $X_w := \{x \in G \mid \varphi_w(x) = x\} = \{x \in G \mid x \preceq_w wx\}$ is a convex subset of G .

Proof. Notice that X_w consists of those $x \in G$ for which the conjugate $x^{-1}wx$ of w is cyclically reduced. Assuming without loss that w is cyclically reduced, we have to show that $y^{-1}wy$ is cyclically reduced whenever $x^{-1}wx$ is cyclically reduced and $y \subset x$. Setting $a = x \cap w, b = x \cap w^{-1}, x' = a^{-1}b^{-1}x$, and $u = a^{-1}wb$, it follows by Lemma 4.11. that $a \bullet x' \cap b \bullet x' = x'$ and $x' \perp u$. Setting $c = y \cap w = y \cap a, d = y \cap w^{-1} = y \cap b, y' = c^{-1}d^{-1}y, \alpha = c^{-1}a, \beta = d^{-1}b$, we get $c \perp d, \alpha \perp d, \beta \perp c$ (by [9, Lemma 2.2.5.] since $a \perp b$), and $x = a \bullet b \bullet x' = c \bullet \alpha \bullet d \bullet \beta \bullet x' = c \bullet d \bullet \alpha \bullet \beta \bullet x'$, so $y' \subset \alpha \bullet \beta \bullet x'$. As $d \bullet y' \cap \alpha = 1$ and $\alpha \perp d$, it follows that $y' \perp \alpha$, and by symmetry, $y' \perp \beta$, so $y' \subset x'$ by (\perp) . Setting $z = y'^{-1}x'$, we get $y' \subset x' \subset a \bullet x' = c \bullet \alpha \bullet y' \bullet z = c \bullet y' \bullet \alpha \bullet z$, therefore $y' \subset c \bullet y'$ by (A_1) , and also, by symmetry, $y' \subset d \bullet y'$. As $c \perp d$ we obtain $c \bullet y' \cap d \bullet y' = y'$ by (A_2) . On the other hand, $y' \perp u$ since $x' \perp u$ and $y' \subset x'$. As we already know that $y' \perp \alpha$ and $y' \perp \beta$ it follows by Corollary 5.6. that $y' \perp \alpha \bullet u \bullet \beta^{-1} (= c^{-1}wd)$, therefore $y^{-1}wy$ is cyclically reduced according to Lemma 4.11. \square

Lemma 6.2. $\varphi_w(G) = X_w$.

Proof. The inclusion $X_w \subseteq \varphi_w(G)$ is trivial. To prove the opposite inclusion, let $x \in G$, and set $w' := x^{-1}wx$. By Lemma 4.3., $w'^2 \cap w'^{-2} = w' \cap w'^{-1}$, therefore $\varphi_w(\varphi_w(x)) =$

$$Y(w\varphi_w(x), \varphi_w(x), w^{-1}\varphi_w(x)) = xY(w'^2 \cap w', w' \cap w'^{-1}, w'^{-1} \cap w'^{-2}) = x(w' \cap w'^{-1}) = \varphi_w(x),$$

i.e. $\varphi_w(x) \in X_w$ for all $x \in G$. \square

Lemma 6.3. *If w is cyclically reduced, then $wx \cap w^{-1}x \subset x$ for all $x \in G$.*

Proof. Setting $a = x \cap w, b = x \cap w^{-1}, y = a^{-1}b^{-1}x$ and $u = a^{-1}wb$, we get as in the initial part of the proof of Lemma 4.11. that $z := wx \cap w^{-1}x = a \bullet u \bullet a \bullet y \cap b \bullet u^{-1} \bullet b \bullet y \subset y$, so $z = a \bullet u \bullet a \bullet z \cap b \bullet u^{-1} \bullet b \bullet z$ by (A_2) . As $a \bullet z \cap b^{-1} = 1$ we get $z \cap w = a \bullet u \bullet a \bullet z \cap b \bullet u^{-1} \bullet b \bullet z \cap a \bullet u \bullet b^{-1} = a \bullet u \cap b \bullet u^{-1} \bullet b \bullet z$, and hence (according to [9, Lemma 2.2.5.]) $z \cap w = z \cap a$, since $a \bullet u \cap b \bullet u^{-1} \bullet b \subset w \cap w^{-2} = 1$ and $z \cap a \bullet u \subset b \bullet u^{-1} \bullet b \bullet z \implies (z \cap a \bullet u) \perp b \bullet u^{-1} \bullet b$. By symmetry, it follows that $z \cap w^{-1} = z \cap b$. Setting $c = z \cap w, d = z \cap w^{-1}, \alpha = c^{-1}a, \beta = d^{-1}b, z' = c^{-1}d^{-1}z$, we get $z = c \bullet d \bullet z' = d \bullet c \bullet z'$, and also $c \perp u$ and $d \perp u$ by [9, Lemma 2.2.5.] As $z = wx \cap w^{-1}x \subset \varphi_w(x)$ it follows by Lemmas 6.1. and 6.2. that $z \in X_w$, i.e. $z^{-1}wz$ is cyclically reduced, and hence $c \bullet z' \cap d \bullet z' = z'$ and $z' \perp \alpha \bullet u \bullet \beta^{-1}$ according to Lemma 4.11. Setting $c' = z'^{-1}cz', d' = z'^{-1}dz'$ and $y' = z^{-1}y$, we get $x = a \bullet b \bullet y = a \bullet b \bullet z \bullet y' = c \bullet \alpha \bullet d \bullet \beta \bullet c \bullet d \bullet z' \bullet y' = c \bullet d \bullet \alpha \bullet \beta \bullet z' \bullet c' \bullet d' \bullet y' = c \bullet d \bullet z' \bullet \alpha \bullet \beta \bullet c' \bullet d' \bullet y'$, therefore $z = c \bullet d \bullet z' \subset x$ as required. \square

Proposition 6.4. *φ_w is a folding of G .*

Proof. We may assume without loss that w is cyclically reduced, i.e. $1 \in X_w$. By Lemmas 6.1. and 6.2., $X_w = \varphi_w(G)$ is a convex subset of G , so it remains to show that $[1, x] \cap X_w = [1, \varphi_w(x)]$ for all $x \in G$. As $\varphi_w(x) = (wx \cap w^{-1}x) \cup (x \cap \varphi_w(x))$, it follows by Lemma 6.3. that $\varphi_w(x) \subset x$, i.e. $\varphi_w(x) \in [1, x]$. Conversely, let $y \in [1, x] \cap X_w$. As $y^{-1}wy$ is cyclically reduced, it follows as above that $y^{-1}\varphi_w(x) = \varphi_{y^{-1}wy}(y^{-1}x) \subset y^{-1}x$, therefore $x^{-1}\varphi_w(x) \subset x^{-1}y \subset x^{-1}$, so $y \subset \varphi_w(x)$, i.e. $y \in [1, \varphi_w(x)]$ as desired. \square

Lemma 6.5. *X_w is closed under the congruence \sim_w . In particular, the folding φ_w induces a folding of the quotient median set G/\sim_w .*

Proof. Let $x \in X_w$ and $y \in G$ be such that $x \sim_w y$, i.e. $[x, wy] = [y, wx]$. Assuming $y \notin X_w$, i.e. $y \notin [wy, w^{-1}y]$, it follows by [7, Corollary 5.2.2.] that there is a prime convex subset P of G such that $wy \in P, w^{-1}y \in P$ and $y \notin P$. As $[x, wy] = [y, wx]$ and $[x, w^{-1}y] = [y, w^{-1}x]$, it follows that $x \notin P, wx \in P$ and $w^{-1}x \in P$, therefore $x \notin [wx, w^{-1}x]$, contrary to the assumption $x \in X_w$. \square

Let \equiv_w denote the negation of the congruence \sim_w in the Heyting algebra $\text{Cong}(G)$, cf. 2.1., and let \ll_w denote the order on G defined by $x \ll_w y$ iff $x \preceq_w y$ and $x \equiv_w y$.

Lemma 6.6. *On $X_w = X_{w^{-1}}$ the preorders \preceq_w and $\preceq_{w^{-1}}$, as well as the orders \ll_w and $\ll_{w^{-1}}$, are opposite.*

Proof. We may assume without loss that $1 \in X_w$ and $x \in X_w$ such that $1 \preceq_w x$, i.e. $x \subset wx$. Let $a = x \cap w, b = x \cap w^{-1}$ and $y = a^{-1}b^{-1}x$. Since w and $x^{-1}wx$ are cyclically reduced, it follows as in the proof of Lemma 4.11. that $a \bullet b \bullet y = x = x \cap wx = a \bullet y$, therefore $b = x \cap w^{-1} = 1$, i.e. $x \preceq_{w^{-1}} 1$ as required. \square

Lemma 6.7. *For all $x \in X_w$, $x \underset{w}{\ll} wx$.*

Proof. As $x \underset{w}{\preceq} wx$ provided $x \in X_w$, it remains to show that $x \in X_w \implies x \underset{w}{\equiv} wx$. Let $y, z \in [x, wx]$ be such that $y \underset{w}{\sim} z$, i.e. $[y, wz] = [z, wy]$. Assuming that $y \neq z$, it follows by [7, Corollary 5.2.2.] that there is a prime convex subset P of G such that $y \in P$ and $z \notin P$, therefore $wy \in P$ and $wz \notin P$. On the other hand, as $y, z \in [x, wx]$, we distinguish the following two cases :

Case (1) : $x \in P$ and $wx \notin P$. As $wy \in [wx, w^2x] \cap P$, we get $w^2x \in P$, therefore $wx \in [x, w^2x] \subseteq P$, i.e. a contradiction.

Case (2) : $x \notin P$ and $wx \in P$. As $wz \in [wx, w^2x] \setminus P$, we get $w^2z \notin P$, and hence $wx \in [x, w^2x] \subseteq G \setminus P$, again a contradiction.

Consequently, $y = z$ as desired. \square

Lemma 6.8. $\varphi_{w^n} = \varphi_w$ for all $n \neq 0$.

Proof. As φ_w and φ_{w^n} are foldings of G it suffices to show that they have a common image, i.e. $X_w = X_{w^n}$. The equality above is now immediate by Corollary 4.4. \square

Lemma 6.9. *For all $x \in G$, $x \underset{w}{\ll} \varphi_w(x)$. In particular, the orders $\underset{w}{\ll}$, $\underset{w^{-1}}{\ll}$ and $\underset{\varphi_w(x)}{\leq}$ coincide on the cell $[x, \varphi_w(x)]$.*

Proof. First let us show that $x \underset{w}{\preceq} \varphi_w(x)$, i.e. $\varphi_w(x) \in [x, w\varphi_w(x)] = [x, \varphi_w(wx)]$. Assuming the contrary, it follows by [7, Corollary 5.2.2.] that there is a prime convex subset P of G such that $x \in P$, $\varphi_w(wx) \in P$, and $\varphi_w(x) \notin P$. Consequently, $wx \notin P$ and $w^2x \in P$, whence $\varphi_w(x) = \varphi_{w^2}(x) = Y(w^2x, x, w^{-2}x) \in P$, i.e. a contradiction.

Next let us show that $x \underset{w}{\equiv} \varphi_w(x)$. Assuming that there are $y, z \in [x, \varphi_w(x)]$ such that $y \neq z$ and $y \underset{w}{\sim} z$, i.e. $[y, wz] = [z, wy]$, it follows by [7, Corollary 5.2.2.] again that there is a prime convex subset P of G such that $y \in P$, $wy \in P$, $z \notin P$, and $wz \notin P$, therefore $\varphi_w(y) \in P$ while $\varphi_w(z) \notin P$, contrary to $\varphi_w(y) = \varphi_w(z) = \varphi_w(x)$ since $y, z \in [x, \varphi_w(x)]$ by assumption. \square

Remarks 6.10. (1) Being a folding, φ_w induces according to 2.2. a quasidirection \bullet_{φ_w} defined by $x \bullet_{\varphi_w} y = Y(x, y, \varphi_w(x))$, whose associated preorder $\underset{\varphi_w}{\preceq}$ is given by $x \underset{\varphi_w}{\preceq} y$ iff $y \bullet_{\varphi_w} x = y$ iff $y \in [x, \varphi_w(y)]$. Notice that $\underset{\varphi_w}{\preceq}$ is finer than the preorders $\underset{w}{\preceq}$ and $\underset{w^{-1}}{\preceq}$.

(2) Obviously, for all $w \in G$, the centralizer $Z_G(w) = \{x \in G | xw = wx\}$ of w in G is contained in the stabilizer $Stab(\varphi_w) = \{x \in G | \varphi_{xwx^{-1}} = \varphi_w\} = \{x \in G | xX_w = X_w\}$ of φ_w under the action from the left of G . However the converse is not necessarily true, as for instance in non-commutative l -groups. Indeed, if G is an l -group, then $\varphi_w = \varphi_1 = 1_G$ for all $w \in G$.

(3) Given a median set X , one assigns to any automorphism s of X the mapping $\varphi_s : X \longrightarrow X$, defined by $\varphi_s(x) = Y(s(x), x, s^{-1}(x))$. According to Proposition 6.4., φ_s is a folding whenever X is the underlying median set of an A -group G and s is the left translation $x \mapsto s(x) := wx$ by some element $w \in G$. An analogous situation is provided by [1, Theorem 6.6.] for a Λ -tree (cf. [21]) X , where Λ is a totally ordered Abelian group,

and a *hyperbolic automorphism* s of X . In this case, the s -axis $X_s := \varphi_s(X)$ is identified with a convex subset of Λ , and $s|_{X_s}$ is equivalent to a translation $x \mapsto x + l(s)$, where $0 < l(s) = \min_{x \in X} d(x, s(x))$ ($= d(p, s(p))$ for some (for all) $p \in X_s$) is the *hyperbolic length* of s .

More generally, we can consider a *faithfully full Λ -metric median set* (cf. [8, 1.3.]), where Λ is an Abelian l -group, and an automorphism s of X . Then $\varphi := \varphi_s$ is an endomorphism of the underlying median set of X satisfying the following equivalent conditions :

- (i) $\varphi^3 = \varphi$ and φ^2 is a folding;
- (ii) $\varphi^3 = \varphi$ and $\varphi(X)$ is a convex subset of X ;
- (iii) $\forall x, y, z \in X, \varphi(Y(x, y, \varphi(z))) = Y(\varphi(x), \varphi(y), z)$.

The following assertions hold :

- (a) $X_s := \varphi(X) = \varphi^2(X) = \{x \in X \mid \varphi^2(x) = x\}$ is a retractible convex subset of X ;
- (b) $\varphi|_{X_s}$ is an involutive automorphism of X_s , and $\varphi|_{X_s} = s|_{X_s} \iff \text{Fix}(s^2) := \{x \in X \mid s^2(x) = x\} \neq \emptyset$;
- (c) $l(s) := (d(x, s^2(x)) - d(x, s(x)))_+ \in \Lambda_+$ does not depend on the element $x \in X$, and $l(s) = 0 \iff \text{Fix}(s^2) \neq \emptyset$;
- (d) $[x, s(x)] = [\varphi(x), \varphi(s(x)) = s(\varphi(x))]$ for all $x \in X_s$;
- (e) $d(x, \varphi(s(x))) = l(s)$ for all $x \in X_s$;
- (f) $\forall x \in X, d(x, s(x)) = d(\varphi(x), \varphi(s(x))) + 2d(x, \varphi^2(x)) = l(s) + d(\varphi(x), \varphi^2(x)) + 2d(x, \varphi^2(x))$.

Details will be given in a forthcoming paper. Notice also that the pair (φ, φ) , with $\varphi = \varphi_s$ as above, is a particular case of the so called *compatible pairs* cf. [12, Section 11] which are a basic ingredient for the construction of *universal coverings* relative to *median groupoids of median sets* and *simplicial median groupoids of groups* [12, Proposition 11.4., Theorem 14.1.], which extend the *universal covering relative to a connected graph of groups* [22, Ch. I, Theorem 12]

7 Quasidirections induced by elements of right-angled Artin groups

In the rest of the paper we assume that (G, S) is a right-angled Artin group. By Corollary 3.7., the partial order \subset induced by the canonical length function on (G, S) makes G a simplicial A -group, so we can apply to this special case the general theory developed in the previous sections.

Lemma 7.1. *For all $w, x \in G$, there exists $y \in G$ such that $x \cap wy = y$.*

Proof. We argue by induction on the length $d := l(x)$. The case $d = 0$ is trivial. Assuming that the equality $x \cap wy = y$ is satisfied for some $y \in G$, let $s \in \tilde{S} = S \cup S^{-1}$ be such that $xs = x \bullet s$. If $xs \cap wy = y$, then we have nothing to prove, so let us assume that $y \subsetneq xs \cap wy$, therefore, by (\perp) , $s \perp y^{-1}x$, $x \bullet s = y \bullet s \bullet (y^{-1}x)$, $wy = y \bullet s \bullet (s^{-1}y^{-1}wy)$, and $xs \cap wy = ys$. We distinguish two cases :

Case (1) : $wys = (wy) \bullet s = y \bullet s \bullet (s^{-1}y^{-1}wy) \bullet s$. As $y^{-1}x \cap s^{-1}y^{-1}wy = 1$ and $y^{-1}x \perp s$, we get $y^{-1}x \cap (s^{-1}y^{-1}wy) \bullet s = 1$, and hence $xs \cap wys = ys$ as required.

Case (2) : $wys \subset wy$. Thus $s \subset y^{-1}w^{-1} = (y^{-1}w^{-1}ys) \bullet s^{-1} \bullet y^{-1}$, therefore $s \subset y^{-1}w^{-1}ys$, since otherwise $s \subset s^{-1} \bullet y^{-1}$ by (\perp) , contrary to (A_4) . Consequently, $xs \cap wys = xs \cap wy = ys$ as desired. \square

Proposition 7.2. *For all $w \in G$, the preorder \preceq_w , defined by $x \preceq_w y \iff y \in [x, wy]$, determines a quasidirection \bullet_w on G .*

Proof. By Proposition 5.4., the preorder \preceq_w is compatible with the arboreal structure on G . Moreover each pair (x, y) of elements in G is bounded above with respect to \preceq_w . Indeed, by Lemma 7.1 applied to the elements $x^{-1}wx$ and $x^{-1}y$, there is $z \in G$ such that $x^{-1}y \cap x^{-1}wxz = z$, i.e. $Y(x, y, wxz) = xz$, therefore $x \preceq_w xz$ and $y \preceq_w xz$.

To conclude that the binary operation \bullet_w , defined by $a \bullet_w b = \bigvee_a U_{a,b}$ with $U_{a,b} = \{x \in [a, b] \mid a \preceq_w x \text{ and } b \preceq_w x\}$, is a quasidirection on G , it suffices to show, according to Lemma 2.4., that $c \preceq_w b \implies a \preceq_w b$ whenever the elements $a, b, c \in G$ satisfy $[a, b] = \{a, c, b\}$, $c \notin \{a, b\}$. Setting $s := c^{-1}a$, $t := c^{-1}b$, and $w' := c^{-1}wc$, it follows by assumption that $s, t \in \tilde{S}$, $s \cap t = 1$, i.e. $s \neq t$, and moreover $s \cup t = \infty$, so either $t = s^{-1}$ or $st \neq ts$. Assuming that $c \preceq_w b$, i.e. $t \subset w't$, it follows that $s \cap w't = 1$, as $s \cup t = \infty$, and hence $t \in [s, w't]$, i.e. $a \preceq_w b$, as required. \square

Proposition 7.3. *Given $a \in X_w := \varphi_w(G)$, let $X_{w,a}$ denote the convex closure of the subset $\{w^n a \mid n \in \mathbb{Z}\}$, and let $\Psi_{w,a}$ denote the folding of G associated to $X_{w,a}$. Then, the following assertions hold.*

- (1) $X_{w,a} = \bigcup_{n \geq 0} [w^{-n}a, w^n a]$ is an unbounded distributive lattice with respect to the order \ll_w , the join $x \bullet_w y = y \bullet_w x$, and the meet $x \bullet_{w^{-1}} y = y \bullet_{w^{-1}} x$ for $x, y \in X_{w,a}$.
- (2) For all $x \in G$, $\Psi_{w,a}(x) = \lim_{n \rightarrow \infty} Y(w^{-n}a, x, w^n a)$, i.e. there exists $m \geq 0$ such that $\Psi_{w,a}(x) = Y(w^{-n}a, x, w^n a)$ for all $n \geq m$.
- (3) X_w is the closure of $X_{w,a}$ under the congruence \sim_w .
- (4) $X_{w,a} = X_w \cap \overset{w}{\equiv} a$.

Proof. (1) Since $w^n a \ll_w w^m a$ for $n, m \in \mathbb{Z}, n \leq m$, by Lemma 6.7., and the orders \ll_w and \leq_y coincide on $[x, y] = \{z \in G \mid x \ll_w z \ll_w y\}$ provided $x \ll_w y$, it follows that $X_{w,a}$ is the union of the ascending chain of cells $[w^{-n}a, w^n a]$ for $n \geq 0$, which is directed by the order \ll_w . As the orders \ll_w and $\ll_{w^{-1}}$ are opposite on X_w according to Lemma 6.6., we obtain the desired structure of distributive lattice on $X_{w,a}$.

(2) Thanks to the definition of $\Psi_{w,a}$ and to (1), for all $x \in G$, there is $m \geq 0$ such that $[a, \Psi_{w,a}(x)] = [a, x] \cap X_{w,a} = [a, x] \cap [w^{-n}a, w^n a] = [a, Y(w^{-n}a, x, w^n a)]$ for all $n \geq m$, therefore $\Psi_{w,a}(x) = \lim_{n \rightarrow \infty} Y(w^{-n}a, x, w^n a)$.

(3) We have to show that for all $x \in X_w$, $x \sim_w \Psi_{w,a}(x)$, i.e. $w^{-n}a \preceq_w x \preceq_w w^n a$ for large enough n . Let $m \geq 0$ be such that $\Psi_{w,a}(x) = Y(w^{-n}a, x, w^n a)$ for $n \geq m$. Assuming that $x \not\preceq_w w^m a$, i.e. $w^m a \notin [x, w^{m+1}a]$, it follows by [7, Corollary 5.2.2.] that there is a prime convex subset P of G such that $[x, w^{m+1}a] \subseteq P$ while $w^m a \notin P$, therefore $\Psi_{w,a}(x) = Y(w^{-m-1}a, x, w^{m+1}a) \in P$. As we also have $\Psi_{w,a}(x) = Y(w^{-m}a, x, w^m a) \in P$, and $w^m a \notin P$, it follows that $w^{-m}a \in P$, and hence $w^m a \in [w^{-m}a, w^{m+1}a] \subseteq P$, i.e. a contradiction. On the other hand, as $x \in X_w$, we may interchange the roles of a and x to get some $k \geq 0$ subject to $a \preceq_w w^k x$, i.e. $w^{-k}a \preceq_w x$. Taking $n = \max(m, k)$, it follows that $w^{-n}a \preceq_w x \preceq_w w^n a$ as required.

(4) The inclusion $X_{w,a} \subseteq X_w \cap \overline{\overline{a^w}}$ is immediate by Lemma 6.7. Conversely, assuming that $x \in X_w \cap \overline{\overline{a^w}}$, it follows by (3) that there is $y \in X_{w,a}$ such that $x \sim_w y$, therefore $x = y \in X_{w,a}$ since we also have $x \equiv_w a \equiv_w y$. \square

Corollary 7.4. *Let $a \in X_w$. Then, X_w is the convex closure of the $Z_G(w)$ -orbit of a .*

Proof. Obviously $ua \in X_w$ for all $u \in Z_G(w)$. If $x \in X_w$, then Proposition 7.3. provides a natural number n and some $y \in [w^{-n}a, w^n a]$ such that $x \sim_w y$, i.e. $x^{-1}y \perp x^{-1}wx$ (by Lemma 5.2.), therefore $xy^{-1} \in Z_G(w)$. It follows that $x \in [ua, va]$ with $u = xy^{-1}w^{-n}$, $v = xy^{-1}w^n \in Z_G(w)$. \square

Corollary 7.5. *Given $w \in G$ and $a \in X_w$, let $\varphi_{w,a}$, resp. $\Psi_{w,a}$, denote the folding of G associated to the convex subset \tilde{a}^w , resp. $X_{w,a}$. Then, the median set morphism $X_w \longrightarrow \tilde{a}^w \times X_{w,a}$, $x \mapsto (\varphi_{w,a}(x), \Psi_{w,a}(x))$ is an isomorphism, whose inverse sends a pair $(y, z) \in \tilde{a}^w \times X_{w,a}$ to $ya^{-1}z = za^{-1}y$.*

Proof. By Proposition 7.3., $X_{w,x} = X_w \cap \overline{\overline{x^w}}$ for all $x \in X_w$, and $X_{w,x} \cap \tilde{y}^w \neq \emptyset$ for all $x, y \in X_w$. Thus we may apply Lemma 2.1. and Corollary 2.2. to conclude that the mapping above is an isomorphism of median sets, whose inverse sends a pair $(y, z) \in \tilde{a}^w \times X_{w,a}$ to $\Psi_{w,y}(z) = \varphi_{w,y}(z)$. As $[y, z] = [\varphi_{w,y}(z), \Psi_{w,y}(z)] = [a, \Psi_{w,y}(z)]$ by Lemma 2.1., we get $\Psi_{w,y}(z) = ya^{-1}z = za^{-1}y$ as required. \square

The next statement provides a description of the quasidirection \bullet_w by means of the folding φ_w .

Corollary 7.6. *Let $w \in G$. Then, $x \bullet_w y = \lim_{n \rightarrow \infty} Y(x, y, w^n \varphi_w(x))$ for all $x, y \in G$.*

Proof. Let $x, y \in G$. As $x \ll_w w^m \varphi_w(x)$ for all $m \geq 0$ by Lemmas 6.7 and 6.9., taking into account the definition of \bullet_w , it suffices to show that $y \preceq_w w^n \varphi_w(x)$ for large enough n . By Lemma 6.9., $y \ll_w \varphi_w(x)$, so, in particular, $y \preceq_w \varphi_w(y)$. On the other hand, according to Proposition 7.3. there exist $m \geq 0$ and $z \in [w^{-m} \varphi_w(x), w^m \varphi_w(x)]$ such that $\varphi_w(y) \sim_w z$, whence $\varphi_w(y) \preceq_w w^n \varphi_w(x)$ for all $n \geq m$, as required. \square

Corollary 7.7. *For all $w \in G$, the folding φ_w , interpreted as a quasidirection through the convex embedding $\text{Fold}(G) \rightarrow \text{Dir}(\text{Fold}(G)) \cong \text{Qdir}(G)$, is the join of the quasidirections \bullet_w and $\bullet_{w^{-1}}$ in the directed median set $\text{Qdir}(G)$.*

Proof. By definition, the join of the quasidirections \bullet_w and $\bullet_{w^{-1}}$ is the quasidirection $\bullet := Y(\bullet_w, \bullet_{w^{-1}}, \bullet_1)$ defined by $x \bullet y = Y(x \bullet_w y, x \bullet_{w^{-1}} y, x)$ for $x, y \in G$. Given $x, y \in G$, it follows by Corollary 7.6. that there is $n \geq 0$ such that $x \bullet_w y = Y(x, y, w^n \varphi_w(x))$ and $x \bullet_{w^{-1}} y = Y(x, y, w^{-n} \varphi_w(x))$. As $Y(w^n \varphi_w(x), w^{-n} \varphi_w(x), x) = \varphi_w(x)$ by Lemmas 6.7. and 6.9., we get $x \bullet y = Y(x, y, Y(w^n \varphi_w(x), w^{-n} \varphi_w(x), x)) = Y(x, y, \varphi_w(x))$, i.e. \bullet is the quasidirection induced by the folding φ_w . \square

Corollary 7.8. *Let $w \in G$. Then, the convex subset of $\text{Fold}(G)$ obtained by intersecting the cell $[\bullet_w, \bullet_{w^{-1}}]$ of $\text{Qdir}(G)$ with the convex subset $\text{Fold}(G)$ of $\text{Qdir}(G)$ consists of those foldings η of G for which $\eta(x) \in X_{w, \varphi_w(x)}$ for all $x \in G$.*

Proof. By definition, the intersection $[\bullet_w, \bullet_{w^{-1}}] \cap \text{Fold}(G)$ consists of the foldings η of G subject to $Y(x, y, \eta(x)) \in [x \bullet_w y, x \bullet_{w^{-1}} y]$ for all $x, y \in G$. Given such a folding η and taking $x \in \eta(G)$ and $y = \varphi_w(x)$, we get $x = \varphi_w(x)$ since $x \bullet_w \varphi_w(x) = x \bullet_{w^{-1}} \varphi_w(x) = \varphi_w(x)$ by Lemma 6.9. Thus $\eta(G) \subseteq X_w$. On the other hand, taking $y = \eta(x)$, we obtain $\eta(x) \in [x \bullet_w \eta(x), x \bullet_{w^{-1}} \eta(x)]$, therefore, by applying φ_w , we get $\eta(x) = \varphi_w(\eta(x)) \in [\varphi_w(x) \bullet_w \eta(x), \varphi_w(x) \bullet_{w^{-1}} \eta(x)] \subseteq X_w \cap \varphi_w(x) \stackrel{\equiv_w}{=} X_{w, \varphi_w(x)}$ (by Proposition 7.3.(4)). Conversely, if the folding η of G satisfies the condition $\eta(x) \in X_{w, \varphi_w(x)}$ for all $x \in G$, then, thanks to Proposition 7.3.(1), for each $x \in G$ there exists $m \geq 0$ such that $\eta(x) \in [w^{-m} \varphi_w(x), w^m \varphi_w(x)]$ for all $n \geq m$, therefore, by Corollary 7.6., $Y(x, y, \eta(x)) \in [x \bullet_w y, x \bullet_{w^{-1}} y]$ for all $y \in G$. \square

Proposition 7.9. *For $w, a \in G$, let $\vee_{w; a}$ denote the direction on G obtained by applying the folding of $\text{Dir}(G)$ induced by the quasidirection \bullet_w to the internal direction \vee_a on G associated to a , and let $\leq_{w; a}$ denote the associated order on G . Then, the following assertions hold.*

(1) *For $x, y \in G$, $x \vee_{w; a} y = \lim_{n \rightarrow \infty} Y(x, y, Y(w^n \varphi_w(x), a, w^n \varphi_w(y)))$. In particular, the directions $\vee_{w; a}$ and $\vee_{w; \varphi_w(a)}$ coincide.*

(2) *The ray from a in the direction $\vee_{w; a}$, namely $[a, \vee_{w; a}) := [a, \vee_{w; a}] \cap G = \{x \in G \mid a \leq_{w; a} x\}$, consists of those $x \in G$ for which $a \ll_w x$; the orders $\leq_{w; a}$ and $\leq_{w; \varphi_w(a)}$, and the opposite of \leq_a coincide on $[a, \vee_{w; a})$ making it a distributive lattice with the meet \vee_a , the join \bullet_w , and the least element a .*

(3) *The mapping $G \rightarrow \text{Dir}(G)$, $a \mapsto \vee_{w; a}$ is a morphism of median sets inducing a convex embedding of $G / \equiv_w \cong X_w / \equiv_w$ into $\text{Dir}(G)$.*

(4) The quasidirection induced by the folding $\Psi_{w,\varphi_w(a)}$ of G associated to the convex subset $X_{w,\varphi_w(a)}$ is the join in the directed median set $Qdir(G)$ of the directions $\bigvee_{w;a}$ and

$$\bigvee_{w^{-1};a}.$$

$$(5) \left(\bigvee_{w;a}, \bigvee_{w^{-1};a} \right) := \left[\bigvee_{w;a}, \bigvee_{w^{-1};a} \right] \cap G = X_{w,\varphi_w(a)}.$$

Proof. (1) By definition, $x \bigvee_{w;a} y = (x \bullet_w y) \bigvee_a (y \bullet_w x)$ for all $x, y \in G$, and hence the equality stated in (1) is immediate by Corollary 7.6. As φ_w is a folding, we get $\bigvee_{w;a} = \bigvee_{w;\varphi_w(a)}$.

(2) As we also have $x \bigvee_{w;a} y = (x \bigvee_a y) \bullet_w x \bullet_w y$, it follows that $a \bigvee_{w;a} x = a \bullet_w x$, therefore $a \leq_w x \iff a \ll_w x$, as desired. For $x, y \in [a, \bigvee_{w;a}]$ we get $(x \bigvee_a y) \bullet_w x = Y(x, y, a \bullet_w x) = Y(x, y, x) = x$, whence $x \bigvee_{w;a} y = x \bullet_w y = y \bullet_w x$. In particular, for $x, y \in [a, \bigvee_{w;a}]$, $x \leq_{w,a} y \iff x \ll_w y \iff x \in [a, y] \iff y \leq_x x$.

(3) The compatibility with the median operations on G and $Dir(G)$ of the mapping $a \mapsto \bigvee_{w;a}$ is obvious from the definition of $\bigvee_{w;a}$. Moreover, for $a, b \in G$ and $D \in Dir(G)$, denoting by \bigvee the direction $Y(\bigvee_{w;a}, \bigvee_{w;b}, D)$, we get $x \bigvee y = (x \bigvee_{w;a} y) \bigvee_D (x \bigvee_{w;b} y) = Y(x \bullet_w y, y \bullet_w x, a \bigvee_D b)$ for all $x, y \in G$, therefore $\bigvee = \bigvee_{w;(a \bigvee_D b)}$. Thus the image of the mor-

phism $a \mapsto \bigvee_{w;a}$ is a convex subset of $Dir(G)$. It remains to show that $\bigvee_{w;a} = \bigvee_{w;b} \iff a \equiv_w b$. Assuming that $\bigvee_{w;a} = \bigvee_{w;b}$, let $c := a \bigvee_{w;a} b$. By (2) we get $a \ll_w c$ and $b \ll_w c$, whence $a \equiv_w c \equiv_w b$. Conversely, as $a \equiv_w b \iff a \bullet_w b = b \bullet_w a$, $a \equiv_w b$ implies $a \bullet_w b \in [a, \bigvee_{w;a}] \cap [b, \bigvee_{w;b}]$ by (2), therefore, by (2) again, $[a \bullet_w b, \bigvee_{w;a}] = \{x \in G \mid a \bullet_w b \ll_w x\} = [a \bullet_w b, \bigvee_{w;b}]$, i.e. $\bigvee_{w;a} = \bigvee_{w;b}$ as required.

(5) By (1) we may assume that $a \in X_w$. For any $x \in (\bigvee_{w;a}, \bigvee_{w^{-1};a})$ we get $x = a \bigvee_x x \in [a \bigvee_{w;a} x, a \bigvee_{w^{-1};a} x] \subseteq X_{w,a}$ since by (1) there is $n \geq 0$ such that $a \bigvee_{w;a} x = Y(a, x, Y(w^n a, a, w^n \varphi_w(x))) \in [a, w^n a]$, and similarly $a \bigvee_{w^{-1};a} x \in [a, w^{-n} a]$. Conversely, as $\bigvee_{w;b} = \bigvee_{w;a}$ for all $b \in X_{w,a} \subseteq \bar{a}^w$ by (3), it suffices to show that $a \in [\bigvee_{w;a}, \bigvee_{w^{-1};a}]$, i.e. $x \bigvee_a y \in [x \bigvee_{w;a} y, x \bigvee_{w^{-1};a} y]$ for all $x, y \in G$. For $x, y \in G$, let $u := \bigvee_{w;a} \{x, y, a\}$ and $v := \bigvee_{w^{-1};a} \{x, y, a\}$. It follows as above that u and v belong to $X_{w,a}$, and $a \in [v, u]$. Consequently, $x \bigvee_a y = Y(x, y, u \bigvee_a v) = (x \bigvee_u y) \bigvee_a (x \bigvee_v y) = (x \bigvee_{w;a} y) \bigvee_a (x \bigvee_{w^{-1};a} y) \in [x \bigvee_{w;a} y, x \bigvee_{w^{-1};a} y]$ as desired.

(4) We may assume that $a \in X_w$ by (1). By definition we have to show that $Y(\bigvee_{w;a}, x, \bigvee_{w^{-1};a}) = \Psi_{w,a}(x)$ for all $x \in G$, i.e. $\Psi_{w,a}(x) \in [\bigvee_{w;a}, \bigvee_{w^{-1};a}] \cap [x, \bigvee_{w;a}] \cap [x, \bigvee_{w^{-1};a}]$. By (5) we get $\Psi_{w,a}(x) \in [\bigvee_{w;a}, \bigvee_{w^{-1};a}]$, while $\Psi_{w,a}(x) \in [a, x] \cap [a \bigvee_{w;a} x, x]$ (since $a \bigvee_{w;a} x \in X_{w,a}$) implies $Y(x, \bigvee_{w;a}, \Psi_{w,a}(x)) = Y(x, \bigvee_{w;a}, Y(a, x, \Psi_{w,a}(x))) = Y(a \bigvee_{w;a} x, x, \Psi_{w,a}(x)) = \Psi_{w,a}(x)$, i.e. $\Psi_{w,a}(x) \in [x, \bigvee_{w;a}]$. Similarly, we obtain $\Psi_{w,a}(x) \in [x, \bigvee_{w^{-1};a}]$. \square

8 Structure theorems for quasidirections, foldings and centralizers

In this last section of the paper we will show that certain invariants (quasidirections, foldings, centralizers) associated to elements of a given right-angled Artin group (G, S) can be conveniently described in terms of the corresponding invariants associated to the so called *primitive* elements of G .

Before defining the primitive elements of G , we prove some useful statements on centralizers.

Lemma 8.1. *Given $x, y, w \in G$ such that $x \perp y, x \subset w$ and $y \subset w$, the necessary and sufficient condition for xy to belong to $Z_G(w)$ is that x and y belong to $Z_G(w)$.*

Proof. The sufficiency part is trivial, so it remains to show that x and y belong to $Z_G(w)$ whenever $xy \in Z_G(w)$. By assumption $x \cup y = x \bullet y = y \bullet x$, and $w = x \bullet y \bullet z = z \bullet y \bullet x$, where $z = x^{-1}y^{-1}w$. We argue by induction on the length $d := l(w)$. Set $u := x \cap z, x' := u^{-1}x$, and $z' := u^{-1}z$. If $u = 1$ then $y \bullet z = z \bullet y$ by Lemma 3.9., and hence $x, y \in Z_G(w)$. Assuming that $u \neq 1$ and simplifying with u , we obtain $u^{-1}w = x' \bullet y \bullet u \bullet z' = z' \bullet y \bullet u \bullet x'$. As $x' \perp z'$ and $x' \perp y$ thanks to [9, Lemma 2.2.5.], it follows by Lemma 3.9. again that $y \bullet x' \bullet u = x' \bullet y \bullet u = y \bullet u \bullet x'$, therefore $x' \bullet u = u \bullet x'$. Since $y \perp u$, we get $u^{-1}w = x \bullet y \bullet z' = z' \bullet y \bullet x$, i.e. $xy \in Z_G(u^{-1}w)$. As $l(u^{-1}w) = d - l(u) < d$, it follows by the induction hypothesis that $y \in Z_G(u^{-1}w)$, and hence $x, y \in Z_G(w)$, since $yu = uy$. \square

Lemma 8.2. *For $x, y \in G$ and $m \geq 1, x^m = y^m$ implies $x = y$.*

Proof. As $\varphi_x(1) = \varphi_{x^m}(1) = \varphi_{y^m}(1) = \varphi_y(1)$ by Lemma 6.8., we may assume without loss that x and y are both cyclically reduced. Setting $z := x \cap y$ and assuming that $z \neq x$, let $s \in \tilde{S}$ be such that $s \subset z^{-1}x$, whence $s \cap z^{-1}y = 1$. As $s \subset (z^{-1}x) \bullet \underbrace{x^{m-1}}_{m \text{ factors}} = (z^{-1}y) \bullet y^{m-1}$, it follows that $s \perp z^{-1}y$, and $s \subset y^{m-1}$ by (\perp) . Since $x \bullet \underbrace{x \bullet \dots \bullet x}_{m \text{ factors}} =$

$y \bullet \underbrace{y \bullet \dots \bullet y}_{m \text{ factors}}$, the number of the s 's in any reduced decomposition of x equals the

corresponding number for y . As $s \subset z^{-1}x$ we necessarily have $s \sqsubset z^{-1}y$, contrary to $s \perp z^{-1}y$, by [9, Lemma 2.2.5.] Consequently, $z = x \subset y$, and hence $x = y$ by symmetry. \square

Corollary 8.3. *For $w \in G$ and $m \neq 0, Z_G(w^m) = Z_G(w)$.*

Proposition 8.4. *For any cyclically reduced element w of G , the canonical isomorphism of median sets $\tilde{1}^w \times X_{w,1} \rightarrow X_w, (x, y) \mapsto x \bullet y = y \bullet x$ provided by Corollary 7.5. (with $a = 1 \in Z_G(w) \subseteq X_w$) induces an isomorphism of median groups $\tilde{1}^w \times H_w \rightarrow Z_G(w)$, where $H_w := Z_G(w) \cap X_{w,1} = Z_G(w) \cap \tilde{1}^w$.*

Proof. By Corollary 5.6., $\tilde{1}^w \subseteq Z_G(w)$ is a convex subgroup of G , so it is the *special* subgroup of G generated by $S_w := \{s \in S \mid s \perp w\}$. It remains to show that H_w is a median subgroup of G . First note that H_w is a subgroup of $Z_G(w)$. Moreover

the convex subset $\overset{=}{1}$ of G is closed under the action from the left of H_w . Indeed, for $x \in Z_G(w)$ and $y \in G$, we get $y \overset{=}{w} 1 \iff x^{-1}y \overset{=}{x^{-1}wx} x^{-1} \iff x^{-1}y \overset{=}{w} x^{-1}$, and $x^{-1} \overset{=}{w} 1 \iff 1 \overset{=}{xwx^{-1}} x \iff 1 \overset{=}{w} x$, as required.

Thus it remains to show that $Y(x, y, z) \in Z_G(w)$ for all $x, y, z \in H_w$. Since $X_{w,1} = \bigcup_{n \geq 0} [w^{-n}, w^n]$ by Proposition 7.3.(1), it follows that there is $n \geq 1$ such that $x, y, z \in [w^{-n}, w^n] \subseteq X_{w,1}$. Consequently, the elements $w^n x, w^n y$ and $w^n z$ belong to $Z_G(w) \cap [1, w^{2n}] \subseteq Z_G(w^{2n}) \cap [1, w^{2n}]$, therefore $w^n Y(x, y, z) = Y(w^n x, w^n y, w^n z) \in Z_G(w^{2n})$ by Lemma 3.10. As $Z_G(w^{2n}) = Z_G(w)$ according to Corollary 8.3., we conclude that $Y(x, y, z) \in Z_G(w)$ as desired. \square

Lemma 8.5. *Let w be a cyclically reduced element of G , and $a \in X_w$, so the conjugate $a^{-1}wa$ of w is cyclically reduced too. Then, the inner group automorphism $x \mapsto a^{-1}xa$ of the group G induces an isomorphism of median groups $H_w \rightarrow H_{a^{-1}wa}$.*

Proof. By Corollary 7.5., the convex subsets $\tilde{1}^w$ and $X_{w,1}$ are orthogonal, i.e. $x \perp y$ provided $x \in \tilde{1}^w$ and $y \in X_{w,1}$, therefore the mapping $x \mapsto a^{-1}xa$ is the identity on H_w whenever $a \in \tilde{1}^w$. Thus we may assume without loss that $a \in X_{w,1}$. First notice that $a^{-1}H_w a = H_{a^{-1}wa} = Z_G(a^{-1}wa) \cap a^{-1}X_{w,1}$. Indeed, for $x \in H_w$ we get $\varphi_w(xa) = x\varphi_w(a) = xa$, i.e. $xa \in X_w$. On the other hand, $a \overset{=}{w} 1$ implies $xa \overset{=}{xwx^{-1}} x$, therefore $xa \overset{=}{w} 1$, so $xa \in X_{w,1}$, since $wx = xw$ and $x \overset{=}{w} 1$.

As $a \in X_{w,1}$ and $b \subset a \implies b^{-1}a \in b^{-1}X_{w,1} = b^{-1}X_{w,b} = X_{b^{-1}wb,1}$, proceeding by induction on the length $d := l(a)$, we are reduced to the case $d = 1$, i.e. $a \in \tilde{S} \cap X_{w,1}$. Since for each $x \in H_w \subseteq X_{w,1}$ there is $n \geq 0$ such that $[1, x] = [w^{-n} \cap x, w^n \cap x]$, and $w^m \cap x \in H_w$ for all $m \in \mathbb{Z}$ by Proposition 8.4., and since $H_{w^m} = H_w$ for all $m \neq 0$ by Corollary 8.3, it remains to show that $a^{-1}xa \subset a^{-1}ya$ whenever $x \subset y \subset w$. As $a \in \tilde{S} \cap X_{w,1}$ we distinguish the following two cases :

Case (1) : $a \subset w$. If $a \cap x = 1$, whence $a^{-1} \perp x$, then we are done by Lemma 4.10., so we may assume that $a \subset x$. Setting $x' := a^{-1}x, y' := x^{-1}y$ and $z := y^{-1}w$, we get $w = a \bullet x' \bullet y' \bullet z = z \bullet a \bullet x' \bullet y' = y' \bullet z \bullet a \bullet x'$, therefore $a \subset z \bullet a$ by (A_1) , and hence $a \subset y' \bullet a$ by (A_1) again. As $w^2 = w \bullet w$, it follows that $a^{-1}xa = x' \bullet a \subset x' \bullet y' \bullet a = a^{-1}ya$ as required.

Case (2) : $a \subset w^{-1}$. As $w^{-1}y \subset w^{-1}x \subset w^{-1}$, we may apply Case (1) to get $a^{-1}w^{-1}ya \subset a^{-1}w^{-1}xa \subset a^{-1}w^{-1}a$, and hence $a^{-1}xa \subset a^{-1}ya$, as desired. \square

Remark 8.6. For a cyclically reduced element w of G and an element $a \in X_w \setminus X_{w,1}$, the group isomorphism $Z_G(w) \rightarrow Z_G(a^{-1}wa), x \mapsto a^{-1}xa$, is not necessarily an isomorphism of median groups. For instance, let $S = \{a, b, c\}$, and let G be given by the presentation $G = \langle S; [a, c] = [b, c] = 1 \rangle$, so $G \cong F_2 \times \mathbb{Z}$. We obtain $Z_G(c) = X_c = G, X_{c,1} = H_c = \langle c \rangle$, and $a \subset ab$, but $a^{-1}aa = a \not\subset ba = a^{-1}(ab)a$.

The following definition is justified by Lemma 8.5.

Definition 8.7. *A non-trivial element w of G is called primitive if for some (for all) $a \in X_w$, the median subgroup $H_{a^{-1}wa} = Z_G(a^{-1}wa) \cap X_{a^{-1}wa,1}$ is cyclic, generated by $a^{-1}wa$.*

In particular, a cyclically reduced element $w \neq 1$ is primitive iff H_w is generated by w . As $X_{xwx^{-1}} = xX_w$ for all $x, w \in G$, the primitiveness is preserved by conjugation.

The next lemma provides equivalent descriptions for primitive elements.

Lemma 8.8. *The following assertions are equivalent for an element $w \neq 1$ of G .*

- (1) w is primitive.
- (2) The cell $C := [1, \varphi_w(1)^{-1}w\varphi_w(1)]$ is quasilinear, i.e. $|\partial C| \leq 2$, and w is not a proper power of some element of G .
- (3) For all $x \in G$, the cell $[1, xwx^{-1}]$ is quasilinear, and w is not a proper power of some element of G .

Proof. (1) \implies (3). Assume that w is primitive, and let $x \in G$. By Lemma 4.2., $xwx^{-1} = u \bullet v \bullet u^{-1}$, where $u = \varphi_{xwx^{-1}}(1) = x\varphi_w(x^{-1})$, and $v = u^{-1}xwx^{-1}u$ is cyclically reduced and primitive. In particular, v (and hence w) cannot be a proper power since assuming $v = v'^n$ for some $v' \in G, n \geq 1$, we get $v' \in H_{v'} = H_v$, and hence $n = \pm 1$. Assuming that $[1, xwx^{-1}] = [a, b]$ for some $a, b \in G$, i.e. $a \perp b$ and $xwx^{-1} = a \bullet b = b \bullet a$, and setting $u_1 := u \cap a, u_2 := u \cap b, a' := u_1^{-1}au_1$ and $b' := u_2^{-1}bu_2$, we obtain $a = u_1 \bullet a' \bullet u_1^{-1}, b = u_2 \bullet b' \bullet u_2^{-1}$, and $[1, v] = [a', b']$, therefore either $a' = 1$ or $b' = 1$ since $a', b' \in [1, v] \cap Z_G(v) = \{1, v\}$. In the former case we get $a = 1$, while in the latter case it follows that $b = 1$. Thus the cell $[1, xwx^{-1}]$ is quasilinear as required.

(3) \implies (2) is trivial.

(2) \implies (1) It suffices to show that $[1, w] \cap Z_G(w) = \{1, w\}$ whenever the cyclically reduced element w is not a proper power and the cell $[1, w]$ is quasilinear. Let $a \in [1, w] \cap Z_G(w)$ be such that $a \neq 1$ and its length $l(a)$ is minimal. We have to show that $a = w$. Let $n \geq 1$ be such that $a^n = \underbrace{a \bullet a \bullet \dots \bullet a}_{n \text{ factors}} \subset w$ and $a^{n+1} \not\subset w$. Setting

$b := a^{-n}w$, we obtain $w = a \bullet b = b \bullet a$. As $Z_G(w)$ is a median subgroup of G we get $a \cap b \in [1, w] \cap Z_G(w)$, therefore $a \cap b = 1$ by the minimality of $l(a)$ and the maximality of n . Consequently, $[1, w] = [a^n, b]$, and hence $w = a^n$ since the cell $[1, w]$ is quasilinear by assumption. As w is not a proper power, we obtain $a = w$ as desired. \square

Let $\text{Prim}(G)$ denote the subset of all primitive elements of G . Obviously, $\tilde{S} \subseteq \text{Prim}(G)$, and $\text{Prim}(G)$ is closed under the operation $w \mapsto w^{-1}$. In particular, if G is freely generated by S then $\text{Prim}(G)$ consists of those $w \in G \setminus \{1\}$ which are not proper powers, while $\text{Prim}(G) = \tilde{S}$ whether G is the free Abelian group generated by S .

The elements of G admit canonical representations as products of powers of commuting primitive elements, as follows.

Theorem 8.9. *For a given element $w \in G$, there exist primitive elements p_1, \dots, p_n and positive integers m_1, \dots, m_n such that $a^{-1}p_i a \perp a^{-1}p_j a$ for $i \neq j$ and $a \in X_w$ (in particular, the p_i 's are commuting primitive elements), and $w = \prod_{i=1}^n p_i^{m_i}$. The pairs (p_i, m_i) are uniquely determined up to a permutation of the indices $i = 1, \dots, n$.*

Proof. For all $a \in X_w$, $[1, a^{-1}wa] \cap Z_G(a^{-1}wa)$ is a median subset of the median group $H_{a^{-1}wa} = Z_G(a^{-1}wa) \cap a^{-1}X_{w,a}$. Given $a \in X_w$, let u_1, \dots, u_n be the minimal elements of $[1, a^{-1}wa] \cap Z_G(a^{-1}wa)$ with respect to the order \subset . Obviously, the u_i 's

are cyclically reduced and pairwise orthogonal. Also they are not proper powers by Corollary 8.3. Moreover, assuming $[1, u_i] = [u'_i, u''_i]$, it follows by Lemma 8.1. that $u'_i \in [1, a^{-1}wa] \cap Z_G(a^{-1}wa)$, therefore $u'_i \in \{1, u_i\}$ by the minimality of u_i . Thus the cells $[1, u_i], i = 1, \dots, n$, are quasilinear, and hence the u'_i s are primitive according to Lemma 8.8. Let $m_i \geq 1$ be the largest natural number for which $u_i^{m_i} \subset a^{-1}wa$, and let $u = \cup_{i=1}^n u_i^{m_i} = \prod_{i=1}^n u_i^{m_i}$ and $v = u^{-1}(a^{-1}wa)$. As $u \in [1, a^{-1}wa] \cap Z_G(a^{-1}wa)$,

we get $v \in [1, a^{-1}wa] \cap Z_G(a^{-1}wa)$, therefore, assuming $v \neq 1$, there is an index i such that $u_i \subset w$. Writing $u = u_i^{m_i} \bullet u'$, with $u_i \perp u'$, and $v = u_i \bullet v'$, we get $a^{-1}wa = u \bullet v = u_i^{m_i} \bullet u' \bullet u_i \bullet v' = u_i^{m_i+1} \bullet u' \bullet v'$, contrary to the definition of m_i . Consequently, $a^{-1}wa = u$. Setting $p_i := au_i a^{-1}$ for $i = 1, \dots, n$, we obtain a representation of $w = \prod_{i=1}^n p_i^{m_i}$ as a product of powers of the commuting primitive elements p_1, \dots, p_n , so to end the proof of the existence part of the statement, it remains to note that $b^{-1}p_i b \perp b^{-1}p_j b$ for $i \neq j$ and $b \in X_w$, since the conjugation map $x \mapsto (a^{-1}b)^{-1}x(a^{-1}b)$, where $a^{-1}b \in a^{-1}X_w = X_{a^{-1}wa}$, induces an isomorphism of median groups $H_{a^{-1}wa} \longrightarrow H_{b^{-1}wb}$ according to Lemma 8.5.

To prove the uniqueness up to permutation, assume that the pairs $(p_i, m_i), i = 1, \dots, n$, satisfy the requirements of the statement. It suffices to show that for all $a \in X_w$, the $a^{-1}p_i a$'s are minimal elements of the lattice $L := [1, a^{-1}wa] \cap Z_G(a^{-1}wa)$. As $u_i := a^{-1}p_i a \neq 1$ belongs to the lattice L by assumption, there is a minimal element v of L such that $v \subset u_i$. Since $u_i^{m_i} \perp u_i^{-m_i}(a^{-1}wa)$ and $v \in Z_G(a^{-1}wa)$, we get $v \in Z_G(u_i^{m_i})$, therefore $v \in [1, u_i] \cap Z_G(u_i)$ by Corollary 8.3., and hence $v = u_i$ since $v \neq 1$ and u_i is cyclically reduced and primitive by assumption. \square

Remark 8.10. A similar result with Theorem 8.9. above is proved in [13, 23] by different methods.

For any $w \in G$, let $\text{Prim}(w) \subseteq \text{Prim}(G)$ denote the finite set $\{p_1, \dots, p_n\}$ of primitive elements uniquely associated to w by Theorem 8.9. Let $\text{Prim}(w)^\sim$ denote the disjoint union of $\text{Prim}(w)$ and $\text{Prim}(w^{-1}) = \text{Prim}(w)^{-1}$. Notice that $\text{Prim}(xwx^{-1}) = x\text{Prim}(w)x^{-1}$ and $\text{Prim}(xwx^{-1})^\sim = x\text{Prim}(w)^\sim x^{-1}$ for all $x \in G$.

For any $w \in G$, we have denoted by S_w the subset of S consisting of those $s \in S$ for which $s \perp w$. In particular, $S_t = \{s \in S \setminus \{t\} \mid st = ts\}$ for all $t \in S$.

The next statement is an immediate consequence of Proposition 8.4. and Theorem 8.9.

Corollary 8.11. (1) For any cyclically reduced element w of G , $S_w = \bigcap_{p \in \text{Prim}(w)} S_p$ generates the convex subgroup $\tilde{1}^w = \bigcap_{p \in \text{Prim}(w)} \tilde{1}^p$, while the median subgroup $H_w =$

$Z_G(w) \cap X_{w,1}$ is Abelian, freely generated by $\text{Prim}(w)$ and contained in the center of $Z_G(w)$.

(2) For any $w \in G$ and $a \in X_w$, $Z_G(w)$ is the direct product of the right-angled Artin group generated by $\bigcap_{p \in \text{Prim}(w)} aS_{a^{-1}pa}a^{-1}$ and the free Abelian group generated by $\text{Prim}(w)$. In particular, $Z_G(w)$ is a right-angled Artin group.

(3) The center $Z(G)$ of G is the free Abelian group generated by the (possibly empty) set $\{s \in S \mid \forall t \in S, st = ts\}$.

Remarks 8.12. (1) It is known [14] that, by contrast with free groups and free Abelian groups, in general, the partially commutative freeness is not transferable to arbitrary subgroups. A graph theoretic transfer criterion for right-angled Artin groups (G, S) , S finite, is given in [20].

(2) A result similar with Corollary 8.11. is proved in [13, 23] by different methods.

Corollary 8.13. *Let $w \in G$. Then, the following assertions hold.*

(1) For all $a \in X_w$, $X_{w,a}$ is the convex closure of the union of its convex subsets $X_{p,a}$ for $p \in \text{Prim}(w)$, and $X_{w,a} \cong \prod_{p \in \text{Prim}(w)} X_{p,a}$.

(2) $X_w = \bigcap_{p \in \text{Prim}(w)} X_p$.

(3) The folding φ_w is obtained by composing the commuting foldings φ_p for $p \in \text{Prim}(w)$.

(4) For $u \in G$, $\varphi_u \leq \varphi_w$, i.e. $X_u \subseteq X_w$, provided $\text{Prim}(w)^\sim \subseteq \text{Prim}(u)^\sim$.

Proof. (1) The inclusion $X_w \subseteq X_p$ for $p \in \text{Prim}(w)$ is immediate by Theorem 8.9. Writing $w = \prod_{p \in \text{Prim}(w)} p^{m_p}$ with $m_p \geq 1$, we get easily

$$[w^{-n}a, w^n a] = [\bigcup_{p \in \text{Prim}(w)} [p^{-nm_p}a, p^{nm_p}a]] \cong \prod_{p \in \text{Prim}(w)} [p^{-nm_p}a, p^{nm_p}a],$$

therefore $X_{w,a} = [\bigcup_{p \in \text{Prim}(w)} X_{p,a}] \cong \prod_{p \in \text{Prim}(w)} X_{p,a}$ for all $a \in X_w$.

(2) Given $x \in \bigcap_{p \in \text{Prim}(w)} X_p$ and $a \in X_w$, it follows by Corollary 7.5. that $[a, x] = [y_p, z_p]$ with $y_p \in \tilde{a}^p$ and $z_p \in X_{p,a}$ for $p \in \text{Prim}(w)$, and hence $[a, x] = [y, z]$, where $y = \bigvee_a \{y_p \mid p \in \text{Prim}(w)\}$ and $z = \bigvee_x \{z_p \mid p \in \text{Prim}(w)\}$, since the negation operator \neg is a median set automorphism of $\partial[a, x]$. As $y \in \bigcap_{p \in \text{Prim}(w)} \tilde{a}^p = \tilde{a}^w$ and $z \in [\bigcup_{p \in \text{Prim}(w)} X_{p,a}] = X_{w,a}$, we obtain $x \in [y, z] \subseteq [\tilde{a}^w \cup X_{w,a}] = X_w$ as desired.

(3) and (4) are immediate consequences of (2). \square

Remark 8.14. The converse of the assertion (4) above is not necessarily true. For instance, if $G = \langle s, t; [s, t] = 1 \rangle \cong \mathbb{Z} \times \mathbb{Z}$ then $\varphi_s = \varphi_t = 1_G$, but $\text{Prim}(s)^\sim = \{s, s^{-1}\} \neq \{t, t^{-1}\} = \text{Prim}(t)^\sim$.

Corollary 8.15. *For all $w \in G$, $\bullet_w = \bigcap_{p \in \text{Prim}(w)} \bullet_p$, i.e. for all $x, y \in G$, $x \preceq_w y \iff x \preceq_p y$ for all $p \in \text{Prim}(w)$.*

Proof. Proceeding by induction on the distance $d := d(x, y)$ we are reduced to the case $d = 1$, i.e. $x^{-1}y = s \in \tilde{S}$. Without loss we may also assume that $x = 1$ and $y = s \in \tilde{S}$. Thus we have to show that $s \subset ws \iff \forall p \in \text{Prim}(w), s \subset ps$. Setting $a := \varphi_w(1)$, we get $w = a \bullet w' \bullet a^{-1}$, where $w' := a^{-1}wa$ is cyclically reduced. By Lemma 6.8. and Theorem 8.9. it suffices to show that for $u', v' \subset w'$ such that $[1, w'] = [u', v']$,

$s \subset ws \iff s \subset us$ and $s \subset vs$, where $u = au'a^{-1}$ and $v = av'a^{-1}$. As $\varphi_w = \varphi_{u'} \circ \varphi_{v'} = \varphi_{v'} \circ \varphi_{u'}$ by Corollary 8.13.(3), we get $\varphi_w(a^{-1}) = 1 \in [\varphi_{u'}(a^{-1}), \varphi_{v'}(a^{-1})] \subseteq [1, a^{-1}]$, and hence $\varphi_{u'}(a^{-1}) \perp \varphi_{v'}(a^{-1})$.

Setting $b := \varphi_{u'}(a^{-1})^{-1}$, $c := \varphi_{v'}(a^{-1})^{-1}$ and $a' := ab^{-1}c^{-1}$, it follows that $b \perp c$, $a = a' \bullet b \bullet c = a' \bullet c \bullet b$, $\varphi_u(1) = a\varphi_{u'}(a^{-1}) = a' \bullet c$, and $\varphi_v(1) = a\varphi_{v'}(a^{-1}) = a' \bullet b$. As $u \cap a^{-1} = u^{-1} \cap a^{-1} = 1$, we also obtain $\varphi_{u'}(a^{-1}) = u' \bullet a^{-1} \cap a^{-1} = u'^{-1} \bullet a^{-1} \cap a^{-1} = u' \bullet a^{-1} \cap u'^{-1} \bullet a^{-1}$, in particular, $b \perp u'$, and similarly $c \perp v'$. Setting $w'' := a'^{-1}wa'$, $u'' := a'^{-1}ua'$ and $v'' := a'^{-1}va'$, it follows that $[1, w''] = [u'', v'']$, $w = a' \bullet w'' \bullet a'^{-1}$, $u = a' \bullet u'' \bullet a'^{-1}$ and $v = a' \bullet v'' \bullet a'^{-1}$. We distinguish the following two cases :

Case (1) : $ws = w \bullet s = a' \bullet w'' \bullet a'^{-1} \bullet s$. Then, $u^{-1} \cap s = a' \bullet u''^{-1} \bullet a'^{-1} \cap s = a' \bullet u''^{-1} \cap s \subset w^{-1} \cap s = 1$, and, similarly, $v^{-1} \cap s = 1$, so $us = u \bullet s$ and $vs = v \bullet s$. Assuming that $s \subset ws = a' \bullet u'' \bullet v'' \bullet a'^{-1} \bullet s$, but $s \not\subset us = a' \bullet u'' \bullet a'^{-1} \bullet s$, we get $s \perp a' \bullet u''$ and hence $us = s \bullet a' \bullet u'' \bullet a'^{-1}$, contrary to the assumption $s \not\subset us$. Consequently, $s \subset us$ and $s \subset vs$ whenever $s \subset ws$. Conversely, assuming $s \subset us$, but $s \not\subset ws$, we get $s \perp a' \bullet u''$, whence $a' \bullet v'' \bullet s \subset ws$. It follows that $s \cap a' \bullet v'' \bullet s = 1$, and hence $s \not\subset vs$, as required.

Case (2) : $s \subset w^{-1} = a' \bullet w''^{-1} \bullet a'^{-1}$, whence $s \subset a' \bullet w''^{-1}$. If $s \subset a'$ then we have nothing to prove, so let us assume that $s \perp a'$ and $s \subset w''^{-1}$. As $[1, w''^{-1}] = [u''^{-1}, v''^{-1}]$, we may assume that $s \subset u''^{-1}$ and $s \perp v''$. Thus $s \perp v$, therefore $s \subset s \bullet v = v \bullet s$. Setting $u'' = u''' \bullet s^{-1}$, we get $us = a' \bullet u''' \bullet a'^{-1}$ and $ws = a' \bullet u''' \bullet v'' \bullet a'^{-1}$, therefore $s \subset us \iff s \subset u''' \iff s \subset ws$ as desired. \square

The next statement provides a classification of the quasidirections \bullet_w for $w \in G$.

Proposition 8.16. *The mapping $w \in G \mapsto \text{Prim}(w)$ induces an antiisomorphism of the ordered set of the quasidirections \bullet_w for $w \in G$ onto the set $\mathcal{F}(G)$ ordered by inclusion, consisting of the finite subsets $F \subseteq \text{Prim}(G)$ satisfying*

- (i) $X_p \cap X_q \neq \emptyset$ for $p, q \in F$, and
- (ii) $a^{-1}pa \perp a^{-1}qa$ for $p, q \in F, p \neq q$, and for some (for all) $a \in X_p \cap X_q$.

Proof. Notice that for any finite subset $F \subseteq \text{Prim}(G)$, the conditions (i) and (ii) above are equivalent with the apparently stronger conditions

- (i') $\bigcap_{p \in F} X_p \neq \emptyset$, and
- (ii') $a^{-1}pa \perp a^{-1}qa$ for $p, q \in F, p \neq q$, and for some (for all) $a \in \bigcap_{u \in F} X_u$.

According to Corollary 8.15., it remains to show that for $w \in G$ and $p \in \text{Prim}(G)$, $p \in \text{Prim}(w)$ whenever the preorder \preceq_p is finer than the preorder \preceq_w . Without loss we may assume that w is cyclically reduced, i.e. $1 \in X_w$. First let us show that $X_w \cap X_p \neq \emptyset$. Since $\varphi_p(1) \preceq_w \varphi_w(\varphi_p(1))$ by Lemma 6.9., it follows by assumption that $\varphi_p(1) \preceq_p \varphi_w(\varphi_p(1))$. On the other hand, as $\varphi_w(\varphi_p(1)) \in [1, \varphi_p(1)]$ and $1 \ll_p \varphi_p(1)$ by Lemma 6.9. again, we get $\varphi_p(1) = \varphi_w(\varphi_p(1)) \in X_w \cap X_p$ as required. Next let us show that $wx \in X_p$ whenever $x \in X_w \cap X_p$. As $x \in X_p \implies \varphi_p(wx) \in [x, wx]$, and $x \in X_w \implies x \ll_w wx$, it follows that $\varphi_p(wx) \ll_w wx$, whence $\varphi_p(wx) \preceq_p wx$. Since

$wx \ll_p \varphi_p(wx)$ by Lemma 6.9., we get $wx = \varphi_p(wx) \in X_p$ as desired. Consequently, $w^n x \in X_p$ for all $n \geq 0$ provided $x \in X_w \cap X_p$. Thus for all $x \in X_w \cap X_p$ and for all $n \geq 0$, the element $(w^n x)^{-1} p (w^n x)$ is a cyclically reduced conjugate of p . Since there are only finitely many cyclically reduced conjugates of p , there is some $n \geq 1$ such that $p \in Z_G(w^n)$, and hence $p \in Z_G(w)$ by Corollary 8.3. As p is primitive, the cell $[1, p]$ is quasilinear by Lemma 8.8., and hence, according to Proposition 8.4., either $p \in 1^w$ or $p \in H_w = Z_G(w) \cap X_{w,1}$. The former case would imply $p \in \tilde{1}^p$, i.e. $p = 1$, a contradiction, so $p \in H_w$, whence $p \in \text{Prim}(w)^\sim$. The assumption $p^{-1} \in \text{Prim}(w)$ would imply $1 \ll_w p^{-1}$, whence $1 \preceq_p p^{-1}$, i.e. $p = 1$, again a contradiction. Consequently, $p \in \text{Prim}(w)$ as required. \square

Corollary 8.17. *The mapping $(w, a) \in G \times G \mapsto (a^p)_{p \in \text{Prim}(w)} \equiv$ induces a bijection of the set of directions $\{\vee_{w;a} \mid (w, a) \in G \times G\}$ onto the disjoint union $\bigsqcup_{F \in \mathcal{F}(G)} G / \bigcap_{p \in F} \equiv_p$.*

Proof. By Propositions 7.9. and 8.16., it suffices to show that for all $w, u, a, b \in G$, the quasidirections \bullet_w and \bullet_u coincide whenever the directions $\vee_{w;a}$ and $\vee_{u;b}$ coincide, since then we get $\vee_{w;a} = \vee_{u;b} \iff \bullet_w = \bullet_u$ and $a \equiv_w b \iff \text{Prim}(w) = \text{Prim}(u)$ and $a \equiv_p b$ for all $p \in \text{Prim}(w)$. Let $w, u, a, b \in G$ be such that $\vee_{w;a} = \vee_{u;b}$. As $\vee_{w;a} = \vee_{w;\varphi_w(a)}$ we may assume from the beginning that $a \in X_w$ and $b \in X_u$. Moreover we may assume that $a = b \in X_w \cap X_u$ since $a \ll_w a \bullet_w b = a \vee_{w;a} b = a \vee_{u;a} b = b \bullet_u a \gg_u b$. Of course we may also assume that $a = b = 1 \in X_w \cap X_u$, so $\{x \in X_{w,1} \mid 1 \ll_w x\} = \{x \in X_{u,1} \mid 1 \ll_u x\}$. In particular, $1 \ll_w u^n \in X_{w,1}$ for all $n \geq 0$. Since there are only finitely many cyclically reduced conjugates of w it follows that u belongs to the cone of positive elements with respect to the order \ll_w of the free abelian group $H_w = Z_G(w) \cap X_{w,1}$ generated by $\text{Prim}(w)$. Consequently, $\text{Prim}(u) \subseteq \text{Prim}(w)$, and hence, by symmetry, $\text{Prim}(w) = \text{Prim}(u)$, therefore $\bullet_w = \bullet_u$ by Proposition 8.16. \square

Corollary 8.18. *For all $w \in G$, $\text{Stab}(\bullet_w) = \text{Stab}(\varphi_w) = \text{Stab}(X_w) = Z_G(w) = \bigcap_{p \in \text{Prim}(w)} Z_G(p)$, and for all $a \in G$, $\text{Stab}(\vee_{w;a}) = \text{Stab}(\Psi_{w,\varphi_w(a)}) = \text{Stab}(X_{w,\varphi_w(a)})$ is the free Abelian group generated by $\text{Prim}(w)$.*

Proof. The inclusions $\bigcap_{p \in \text{Prim}(w)} Z_G(p) \subseteq Z_G(w) \subseteq \text{Stab}(\bullet_w) \cap \text{Stab}(\varphi_w)$ are trivial. Assuming $x \in \text{Stab}(\bullet_w)$, i.e. $\bullet_{xwx^{-1}} = \bullet_w$, it follows by Proposition 8.16. that $\text{Prim}(xwx^{-1}) = x \text{Prim}(w)x^{-1} = \text{Prim}(w)$, so $x^n p x^{-n} = p$ for $p \in \text{Prim}(w)$ and a divisor n of $|\text{Prim}(w)|$, therefore $p \in Z_G(x^n) = Z_G(x)$ for all $p \in \text{Prim}(w)$, i.e. $x \in \bigcap_{p \in \text{Prim}(w)} Z_G(p)$ as desired. Assuming that $x \in \text{Stab}(\varphi_w) = \text{Stab}(X_w)$, i.e. $xX_w = X_w$, and taking some $a \in X_w$, we obtain a family $(a^{-1}x^{-n}wx^na)_{n \in \mathbb{Z}}$ of cyclically reduced conjugates of w , and hence $x \in Z_G(w)$ by the finiteness argument used in the proof of Corollary 8.17..

On the other hand, it follows by Corollary 8.17. that $Stab(\bigvee_{w;a}) = \{x \in G \mid \bigvee_{xwx^{-1};xa} = \bigvee_{w;a}\} = \{x \in Z_G(w) \mid x\varphi_w(a) \equiv_w \varphi_w(a)\}$ = the free abelian group generated by $Prim(w)$. We get a similar result for $Stab(\Psi_{w,\varphi_w(a)})$ since one checks easily, as in the proof of Corollary 8.17., that for all $w, u \in G, a \in X_w$ and $b \in X_u, \Psi_{w,a} = \Psi_{u,b} \iff Prim(w)^\sim = Prim(u)^\sim$ and $a \equiv_w b$. \square

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